Game Theory:
Basic Notions

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1 Games: examples and definitions

Game theory deals with situations in which multiple rational, self-interested entities (individuals, firms, nations, etc.) have to interact.

A normal-form game tries to model a situation in which the entities have to take their decisions simultaneously and independently.

An example is the following Rock-Paper-Scissors game. We can represent it by a table in which the rows correspond to decisions of Player 1, and the columns to decisions of Player 2.

P1, P2	rock	paper	scissors
rock	draw	P2 wins	P1 wins
paper	P1 wins	draw	P2 wins
scissors	P2 wins	P1 wins	draw

Definition 1.1. A *normal form game* is given by:

- a set N (set of *players*); often we use $N = \{1, 2, \ldots, n\}$
- for each $i \in N$, a nonempty set S_i (*strategies* of player i)

The set $S := S_1 \times S_2 \times \ldots \times S_n$ is called the set of *states* of the game.

- for each $i \in N$, a function $u_i : S \to \mathbb{R}$ (*utility* or *payoff function*)

Example 1.2 (Rock-Paper-Scissors).

u_1, u_2	rock	paper	scissors
rock	0, 0	-1, 1	1, -1
paper	1, -1	0, 0	-1, 1
scissors	-1, 1	1, -1	0, 0

Notice that Rock-Paper-Scissors is a *zero-sum* game: in any state of the game, the sum of the utilities of the players is constant. The Rock-Paper-Scissors game is also *finite*: the set N of players has finite cardinality, as do the strategy sets $S_1, \ldots, S_n$.

Example 1.3 (Prisoner's dilemma). Two suspects are interrogated in separate rooms. Each of them can confess or not confess their crime. If both confess, they get 4 years each in prison. If one confess and the other does not, the one that confessed gets 1 year and the other 5. If both are silent, they get 2 years each.

Like in this case, sometimes it is more natural to use *cost* functions $(c_i)_{i\in N}$ instead of utility functions $(u_i)_{i\in N}$; it is equivalent, since we can always define $u_i := -c_i$.

c_1, c_2	confess	silent
confess	4, 4	1, 5
silent	5, 1	2, 2

Notice that the Prisoner's dilemma is *not* a zero-sum game; however it is a finite game.

So far we saw two-player games, but obviously there are games with more players.

Example 1.4 (Bandwidth sharing). A group of n users has to share a common Internet connection with finite bandwidth. Each user can decide what fraction of the bandwidth to use (any amount between none and all). The payoff of each user is higher if this fraction is higher, but is lower if the remaining available bandwidth is too small (packets get delayed too much).

We can model this by defining

- $N := \{1, \ldots, n\}$;
- $S_i := [0, 1]$ for each $i \in N$;
- $u_i(s) := s_i \cdot (1 - \sum_{j\in N} s_j)$, where $s_i \in S_i$ is the strategy selected by player i and $s = (s_1, s_2, \ldots, s_n)$.

Notice that this game is not finite: the set of players is finite, but the strategy sets have infinite cardinality.

Example 1.5 ("Chicken"). Two drivers are headed against each other on a single lane road. Each of them can continue straight ahead or deviate. If both deviate, they both get low payoff. If one deviates while the other continues, he is a "Chicken" and will get low payoff, while the payoff for the other player will be high. If both continue straight ahead, however, a disaster will occur which will cost a lot to the players, as both cars will be destroyed.

u_1, u_2	deviate	straight
deviate	0, 0	-1, 5
straight	5, -1	-100, -100

Notice that the type of games we discussed (normal-form games) are "one-shot" in the sense that players move simultaneously and interact only once. There are also model of games in which players move one after the other (*extensive games*) or in which the same game is played many times (*repeated games*). However, in the course we will focus on normal-form games.

2 Solution concepts

After we have modeled a game, we would like to know which states of the game represent outcomes that are likely to occur, assuming that players are self-interested and rational. There are different ways to do this; each of them gives rise to a different *solution concept*. Different solution concepts have different interpretations, advantages and drawbacks.

2.1 Dominant strategy equilibrium

Consider a state of a game $\Gamma = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$. The utility of a player i in state $s \in S$ will depend on both the action of player i himself (s_i) as well as on the actions of the other players, which we denote conventionally by s_{-i}. So we can rewrite $u_i(s)$ (utility of player i in state s) as $u_i(s_i, s_{-i})$. *Be careful* when reading (or using) this notation: we are not reordering the components of the vector s, we are just writing them differently. For example, with (z_i, s_{-i}) we simply mean the state vector that is obtained from s by replacing the i-th component of state s with z_i.

The idea of a dominant strategy equilibrium is that if a player has an action that is the best among his actions *independently of what the other players do*, then this is certainly a possible outcome of the game. This is formalized as follows.

Definition 2.1. State $s \in S$ is a *dominant strategy equilibrium* if for all $i \in N$ and for all $s' \in S$,

$$u_i(s_i, s'_{-i}) \geq u_i(s'_i, s'_{-i}).$$

(In terms of costs: $c_i(s_i, s'_{-i}) \leq c_i(s'_i, s'_{-i})$.)

Example 2.2 (Dominant strategy in the Prisoner's dilemma)**.** Is (silent,silent) a dominant strategy in the Prisoner's dilemma game? The answer is no: if $s =$(silent,silent), there is a player ($i = 1$) and there is an alternative state $s' =$(confess,silent) for which $c_1(\text{silent, silent}) > c_1(\text{confess, silent})$. This contradicts the definition.

Is (confess,confess) a dominant strategy? We have to check 8 cases (2 players times 4 states) to be sure, but the answer is yes. The point is that no matter what the other player is doing, for each player it is cheaper to confess. So (confess,confess) is a dominant strategy.

A dominant strategy equilibrium represents a "strong" type of equilibrium: every player can rely on his strategy independently of what the others are doing. Unfortunately, it has a big drawback: it does not always exist!

Exercise 2.1. Show that the Chicken game has no dominant strategy equilibrium.

Since it does not always exist, we cannot use the dominant strategy equilibrium concept to predict what will happen in a game : the players will certainly do *something*, and this something will not in general be a dominant strategy equilibrium, simply because the game might not admit one.

2.2 Pure Nash equilibrium

The idea of a pure Nash equilibrium is of that of calling a state an equilibrium if for every player, *assuming that other players are not changing their action*, the player is selecting his "best" action.

That is, no player has an incentive to deviate unilaterally from his action; no one has an interest to alter the "status quo".

Definition 2.3. A state $s \in S$ is a *pure Nash equilibrium* (PNE) if for all $i \in N$ and for all $s'_i \in S_i$,

$$u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i}).$$

The definition is superficially very similar to that of dominant strategy: take your time to appreciate the difference.

However, there is a similarity and in fact every dominant strategy equilibrium is also a pure Nash equilibrium (can you see why?).

The converse is not true: some games without dominant strategy equilibriums have pure Nash equilibria.

Example 2.4 (PNE in the Chicken game)**.** Is the state $s =$(straight,straight) a PNE in the Chicken game? The answer is no: there is a player ($i = 1$) and an alternative strategy s'_i (deviate) such that $-1 = u_1$(straight,straight) $< u_1$(deviate,straight) $= -100$. This contradicts the definition.

Is the state $s =$(deviate,straight) a PNE in the Chicken game? Let's see. If player 1 knows that player 2 is going straight, deviating (-1) is better than going straight (-100). On the other hand, if player 2 knows that player 1 is deviating, going straight (5) is better than deviating (0). So (deviate,straight) is a PNE.

Notice that PNE need not be unique: in fact, in the Chicken game, there are two PNE (which is the other one?).

Let's look at a more complicated example.

Example 2.5 (PNE in the Bandwidth sharing game)**.** Let's see what player i will do when the strategies of the other players are $s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n$. Let's define $t := \sum_{j \neq i} s_i$. From the point of view of player i, the quantity t is a constant. By definition of the payoffs we have $u_i(s) = s_i \cdot (1 - t - s_i)$. Player i can control the one-dimensional variable $s_i \in [0, 1]$. If we take the derivative of $u_i(s)$ with respect to s_i we obtain

$$\frac{\partial}{\partial s_i} u_i(s) = 1 - t - 2s_i.$$

By standard analysis we know that the maximum of u_i is achieved when $\frac{\partial}{\partial s_i} u_i(s) = 0$ (or, possibly, when s_i is at an extreme point of $[0, 1]$, but this is not the case in our example because we get the worst possible payoff in that case). So the player will select $s_i = \frac{1}{2}(1 - t) = \frac{1}{2}(1 - \sum_{j \neq i} s_j)$. This will be true for all $i \in N$, so by symmetry we find out that $s_i = 1/(n + 1)$ for all i.

Unfortunately, although the PNE solution concept applies to a larger class of games, it has basically the same problem as that of a dominant strategy equilibrium: it does not always exist.

Exercise 2.2. Show that the Rock-Paper-Scissors game has no PNE.

2.3 Mixed Nash equilibrium

So far there was no way for a player to interpolate between two actions: either he selects action s_i or he performs another action s_j. We now relax this constraint by allowing the player to choose actions with certain probabilities. For example he might choose action s_1 with probability 1/4, action s_2 with probability 1/3, and action s_3 with probability 5/12. Such strategies are called *mixed*, in contrast with the usual deterministic *pure* strategies. Pure strategies are perhaps more natural, but often the strategies arising in a game are in fact mixed strategies.

Definition 2.6. A *mixed strategy* for player i is a probability distribution on the set of S_i of pure strategies. That is, it is a function $p_i : S_i \to [0,1]$ such that $\sum_{s_i \in S_i} p_i(s_i) = 1$. A *mixed state* is a family $(p_i)_{i\in N}$ consisting of one mixed strategy for each player.

Notice that every pure state s has probability $p(s) := p_1(s_1) \cdot p_2(s_2) \cdot \ldots \cdot p_n(s_n)$ of being realized.

Thus, a mixed state $(p_i)_{i\in N}$ induces an *expected payoff* for player i equal to $\sum_{s\in S} p(s) \cdot u_i(s)$. This is the expected payoff of a state selected probabilitically by the players according to their mixed strategies.

We can now define the notion of mixed Nash equilibrium (MNE).

Definition 2.7. A mixed state is a *mixed Nash equilibrium* if no player can unilaterally improve his expected payoff by switching to a different mixed strategy.

Since mixed strategies generalize pure strategies, it is not hard to see that every PNE is also a MNE. The opposite is not true. In fact, there are games without PNE that admit MNE. More than that: the surprising fact is that any finite game (game where N and S are finite) admits at least one mixed Nash equilibrium!

Theorem 2.1 (Nash 1950)**.** *Every finite game admits at least one mixed Nash equilibrium.*

Example 2.8 (MNE for the Rock-Paper-Scissors game)**.** We saw that the Rock-Paper-Scissors game has no pure Nash equilibria. According to Nash's Theorem it should have at least one equilibrium. In fact, we claim that if we define $p := (1/3, 1/3, 1/3)$, then (p, p) is a MNE.

Let's verify this. Consider for example player 1. We should check that when player 2 uses probability distribution p, player 1 has no incentive to play a mixed strategy different from p. (We should also do a similar check with the roles of the players reversed, but in this case everything will be symmetric.)

If player 2 uses mixed strategy p, and player 1 uses a generic mixed strategy $q = (a, b, c)$ where $a + b + c = 1$, then the expected payoff for player 1 becomes

$$\begin{aligned}
&a \cdot 1/3 \cdot (0) + a \cdot 1/3 \cdot (-1) + a \cdot 1/3 \cdot (+1) + \\
&b \cdot 1/3 \cdot (+1) + b \cdot 1/3 \cdot (0) + b \cdot 1/3 \cdot (-1) + \\
&c \cdot 1/3 \cdot (-1) + c \cdot 1/3 \cdot (+1) + c \cdot 1/3 \cdot (0) = 0.
\end{aligned}$$

So the expected payoff is a constant (0) no matter what a, b and c are! This means that there is no point for player 1 in changing them. Similarly, when player 1 plays $(1/3, 1/3, 1/3)$, player 2 has no incentive to change his strategy from $(1/3, 1/3, 1/3)$. The two players "lock" each other in the mixed Nash equilibrium.

At this point you might wonder why another mixed state, like $(1/2, 1/4, 1/4)$ for both players, is not a MNE. The reason is that if e.g. player 2 plays something different from $(1/3, 1/3, 1/3)$, then the player 1 is no longer indifferent between his possible responses. In this case, when player 2 plays $(1/2, 1/4, 1/4)$, it will be more convenient for player 1 to play $(0, 1, 0)$ than to play $(1/2, 1/4, 1/4)$: since player 2 is playing Rock more often than Paper or Scissors, it is best for player 1 to always play Paper (you can check this by computing the expected payoff for player 1). So $((1/2, 1/4, 1/4), (1/2, 1/4, 1/4))$ is not a MNE.