Submodular Functions and Discrete Convexity

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Submodular Functions

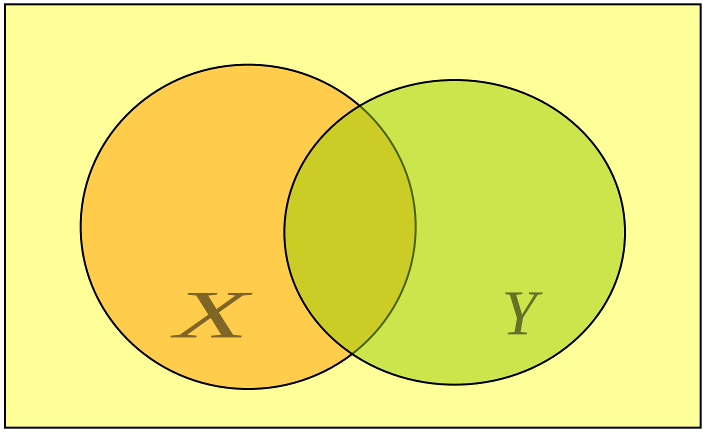
V : Finite Set

$f : 2^V \to \mathbb{R}$

$\forall X, Y \subseteq V$

$$f(X) + f(Y) \geq f(X \cap Y) + f(X \cup Y)$$

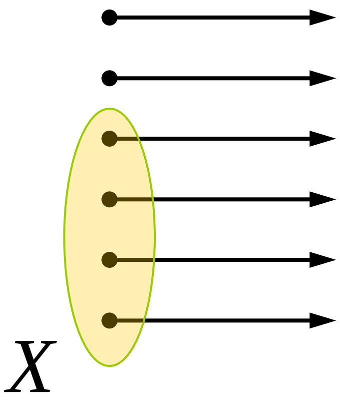
- Cut Capacity Functions
- Matroid Rank Functions
- Entropy Functions



V

Entropy Functions

Information Sources



$$h(\phi) = 0$$

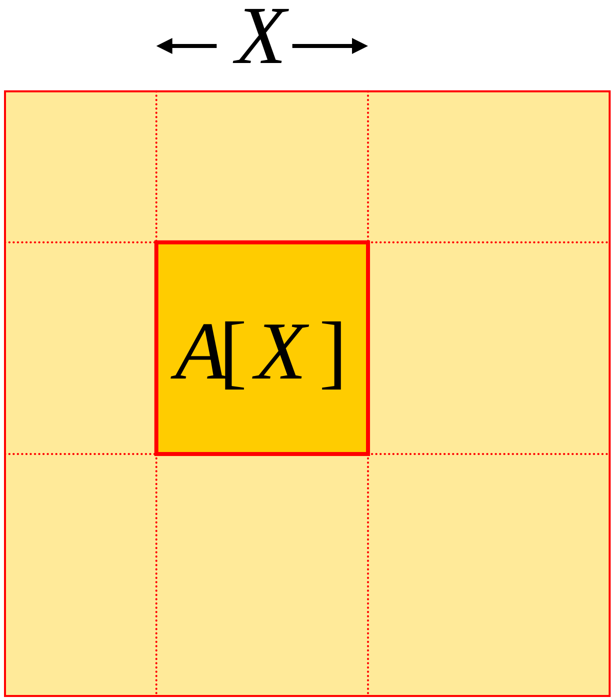
$h(X)$: Entropy of the Joint Distribution

$$h(X) + h(Y) \geq h(X \cap Y) + h(X \cup Y)$$

Conditional Mutual Information ≥ 0

Positive Definite Symmetric Matrices

$A =$



$$f(\phi) = 0$$

$$f(X) = \log \det A[X]$$

Ky Fan's Inequality

$$f(X) + f(Y) \geq f(X \cap Y) + f(X \cup Y)$$

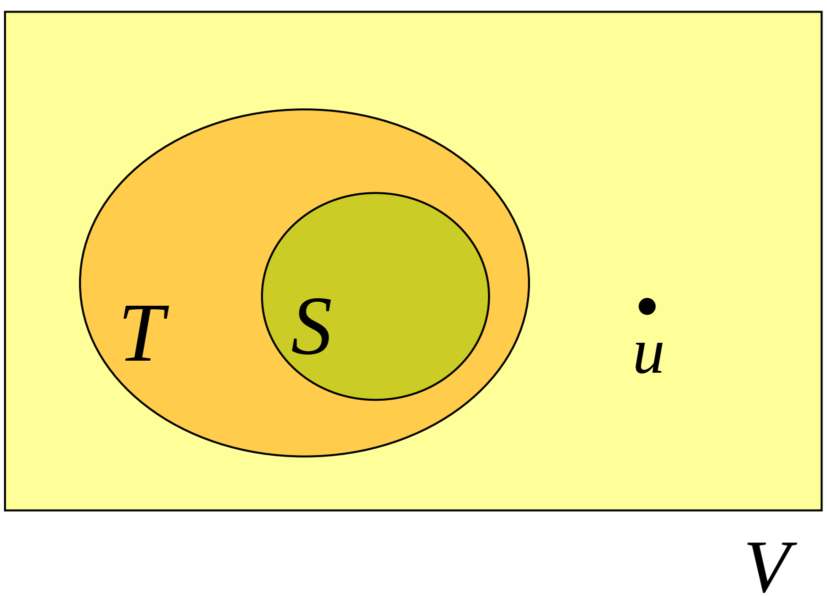
Extension of the Hadamard Inequality

$$\det A \leq \prod_{i \in V} A_{ii}$$

Discrete Concavity

$$S \subseteq T \Rightarrow$$

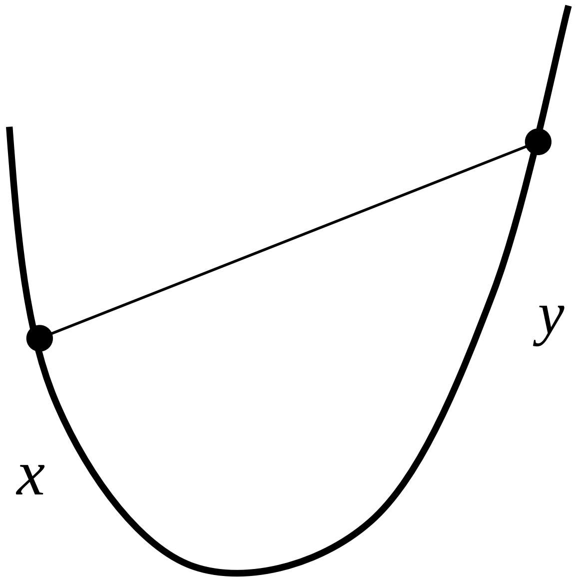
$$f(S \cup \{u\}) - f(S) \geq f(T \cup \{u\}) - f(T)$$



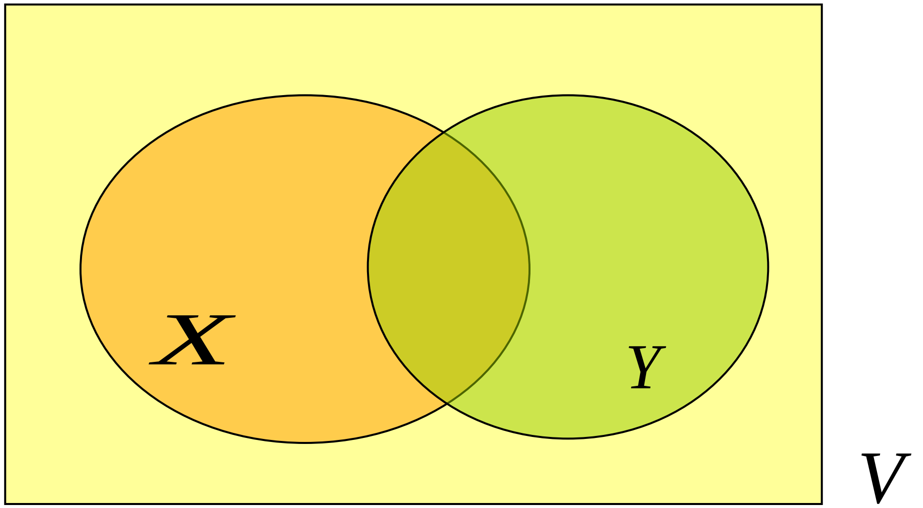
Diminishing Returns

Discrete Convexity

Convex Function

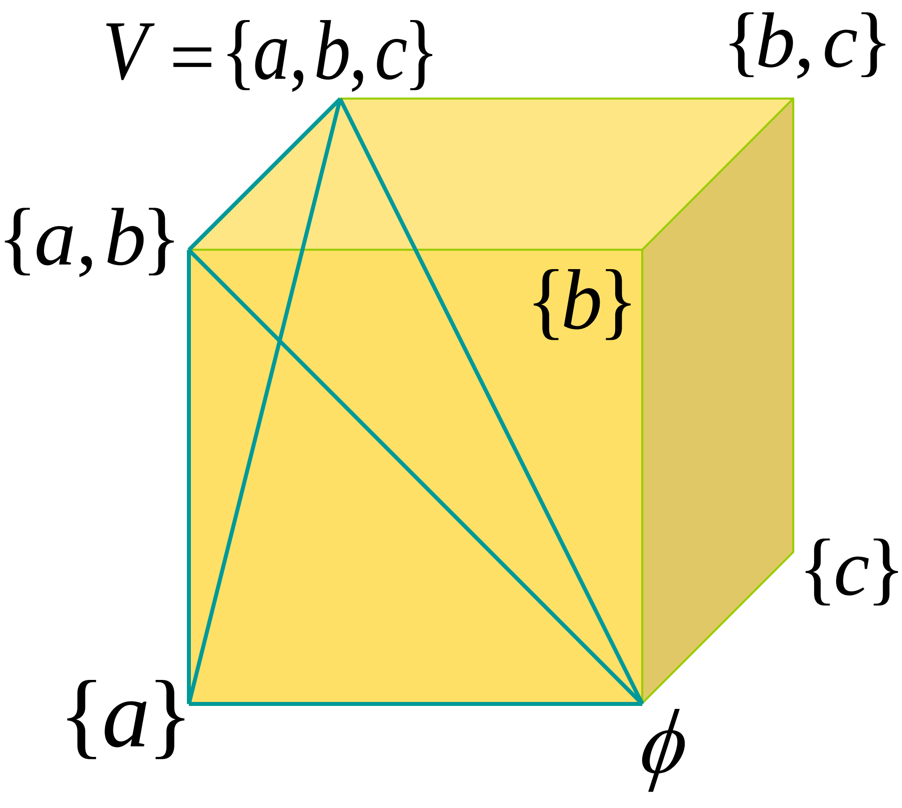


$$f(X)+f(Y)\geq f(X\cap Y)+f(X\cup Y)$$



Discrete Convexity

Lovász (1983)



$\hat{f}$: Linear Interpolation

$\hat{f}$: Convex

⇕

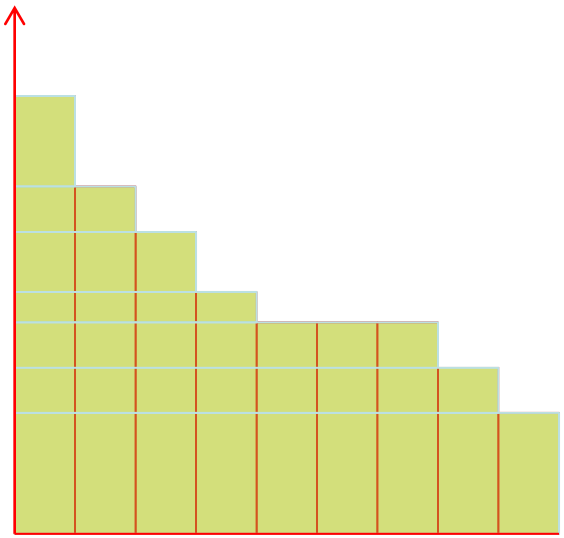
f : Submodular

Discrete Convexity

$$p \in \mathbf{R}_+^V \qquad\qquad f(\phi) = 0$$

$$\hat{f}(p) := \sum_{i=1}^{n} \lambda_i f(S_i)$$

$$\begin{cases} \phi = S_0 \subset S_1 \subset \cdots \subset S_n = V \\ p = \sum_{i=1} \lambda_i \chi_{S_i} \\ \lambda_j \geq 0 \ \ (j = 1, \ldots, n) \end{cases}$$



$$p(v_1) \geq p(v_2) \geq \cdots \geq p(v_n)$$

$$\lambda_j := p(v_j) - p(v_{j+1})$$

Discrete Convexity

$\hat{f}$: Convex ⇨ f : Submodular

$$f(X) + f(Y) = \hat{f}(\chi_X) + \hat{f}(\chi_Y)$$

$$\geq 2\hat{f}\left(\frac{\chi_X + \chi_Y}{2}\right) = f(X \cap Y) + f(X \cup Y).$$

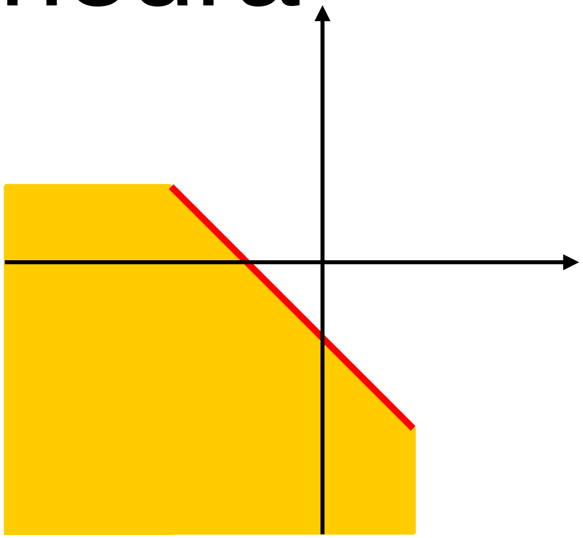
$$\frac{\chi_X + \chi_Y}{2} = \frac{1}{2}\chi_{X \cap Y} + \frac{1}{2}\chi_{X \cup Y}$$

Submodular Polyhedra

$$\mathbf{R}^V = \{x \mid V \to \mathbf{R}\}$$

$$x(Y) = \sum_{v \in Y} x(v)$$

$$f(\phi) = 0$$



Submodular Polyhedron

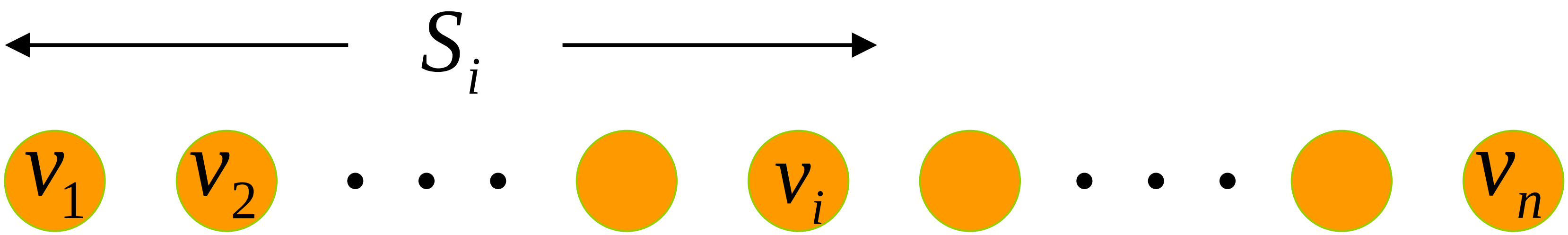
$$P(f) = \{x \mid x \in \mathbf{R}^V, \forall Y \subseteq V, x(Y) \leq f(Y)\}$$

Base Polyhedron

$$B(f) = \{x \mid x \in P(f), x(V) = f(V)\}$$

Greedy Algorithm

Edmonds (1970)
Shapley (1971)



$$y(v_i) = f(S_i) - f(S_{i-1})$$

$$S_0 = \phi$$

$$S_n = V$$

y : Extreme Base

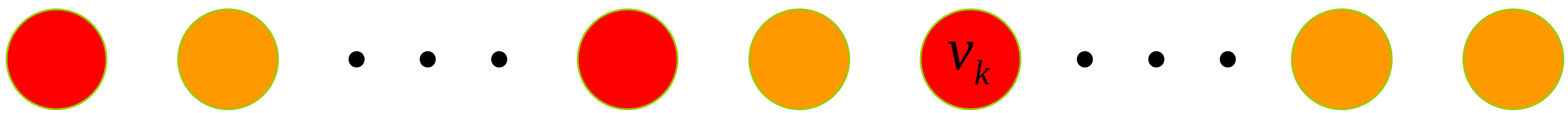
$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} y(v_1) \\ y(v_2) \\ \vdots \\ y(v_n) \end{bmatrix} = \begin{bmatrix} f(S_1) \\ f(S_2) \\ \vdots \\ f(S_n) \end{bmatrix}$$

Greedy Algorithm

$y \in B(f)?$

$$y(X) \le f(X), \quad \forall X \subseteq V$$

Induction on $|X|$



$$y(X \setminus \{v_k\}) \le f(X \setminus \{v_k\})$$

$$\begin{aligned} y(X) &= y(X \setminus \{v_k\}) + y(v_k) \\ &\le f(X \setminus \{v_k\}) + f(S_k) - f(S_{k-1}) \\ &\le f(X) \end{aligned}$$

Submodularity

Linear Optimization

$p \in \mathbf{R}_{+}^{V}$

Edmonds (1970)

$$\max\{\langle p, x\rangle \mid x \in P(f)\}?$$

$$\langle p, x\rangle := \sum_{v \in V} p(v)x(v)$$

Greedy Algorithm with $p(v_1) \geq p(v_2) \geq \cdots \geq p(v_n)$

$$y(v_i) = f(S_i) - f(S_{i-1}) \qquad S_i = \{v_1, \ldots, v_i\}$$

$$\langle p, y\rangle = \sum_{i=1}^{n} p(v_i)[f(S_i) - f(S_{i-1})]$$

$$= \sum_{j=1}^{n} [p(v_j) - p(v_{j+1})] f(S_j) = \hat{f}(p)$$

Linear Optimization

Dual LP

Minimize $\displaystyle\sum_{X \subseteq V} q_X f(X)$

subject to $\displaystyle\sum_{X \ni v} q_X = p(v), \qquad \forall v \in V,$

$q_X \geq 0, \qquad \forall X \subseteq V.$

$$q_X := \begin{cases} p(v_j) - p(v_{j+1}) & (X = S_j) \\ 0 & (\text{otherwise}) \end{cases}$$

$$\sum_{X \subseteq V} q_X f(X) = \sum_{j=1}^{n} [p(v_j) - p(v_{j+1})] f(S_j) = \hat{f}(p)$$

Discrete Convexity

f : Submodular

↓

$$\hat{f}(p) = \max\{\langle p, x\rangle \mid x \in P(f)\}$$

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$\hat{f}$: Convex