Submodular Functions in Graph Theory

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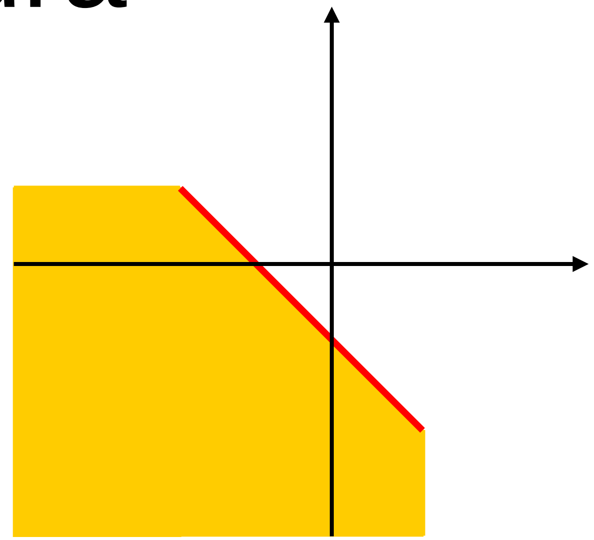
Base Polyhedra

$$\mathbf{R}^V = \{x \mid V \to \mathbf{R}\}$$



$$x(Y) = \sum_{v \in Y} x(v)$$

$$f(\phi) = 0$$

Submodular Polyhedron

$$P(f) = \{x \mid x \in \mathbf{R}^V, \forall Y \subseteq V, x(Y) \le f(Y)\}$$

Base Polyhedron

$$B(f) = \{x \mid x \in P(f), x(V) = f(V)\}$$

Tight Sets

$x \in P(f)$

Y : Tight $\quad\quad x(Y) = f(Y)$

$Y, Z \subseteq V$: Tight $\Rightarrow$ $Y \cup Z, Y \cap Z$: Tight

$\because$)

$$x(Y) + x(Z) = x(Y \cup Z) + x(Y \cap Z)$$

$$f(Y) + f(Z) \geq f(Y \cup Z) + f(Y \cap Z)$$

Upper Base

$$x \in P(f) \Rightarrow \exists h \in B(f),\ h \geq x.$$

$\because$) $D(x)$: Unique Maximal Tight Set

$v \in V \setminus D(x),$

$$\alpha := \min\{ f(Y) - x(Y) \mid v \in Y \}$$

$$x' := x + \alpha \chi_v \in P(f),\ v \in D(x').$$

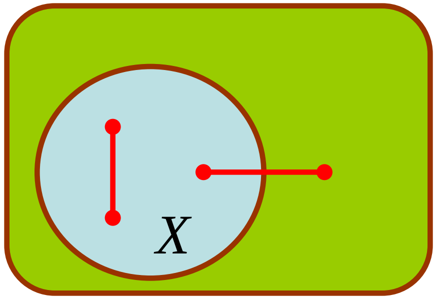
$$f : 2^V \to \mathbf{Z},\ x \in P(f) \cap \mathbf{Z}^V$$

$$\Rightarrow \exists h \in B(f) \cap \mathbf{Z}^V,\ \ h \geq x$$

Graph Orientation

$G=(V,E)$: Graph $\quad b:V\to\mathbf{Z}_+$

$e(X)$: Number of Edges Incident to X.



e : Submodular

Hakimi [1965]

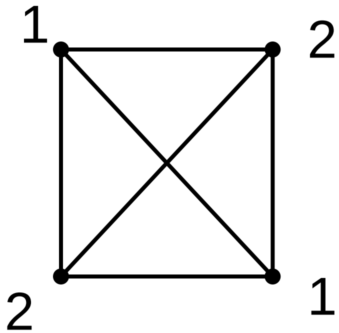
There exists an orientation $\vec{G}$ with in-deg$(v)=b(v)$ for every $v\in V$.

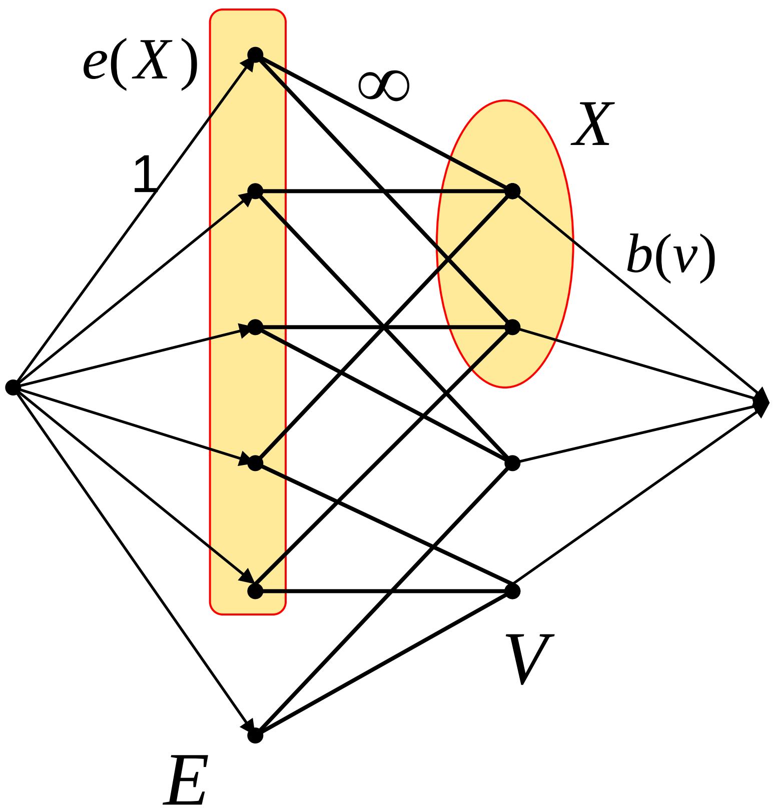
$\Longleftrightarrow$

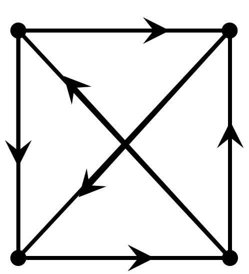
$$b(X)\le e(X),\quad \forall X\subseteq V,$$
$$b(V)=e(V).$$

$b\in B(e)$

Graph Orientation







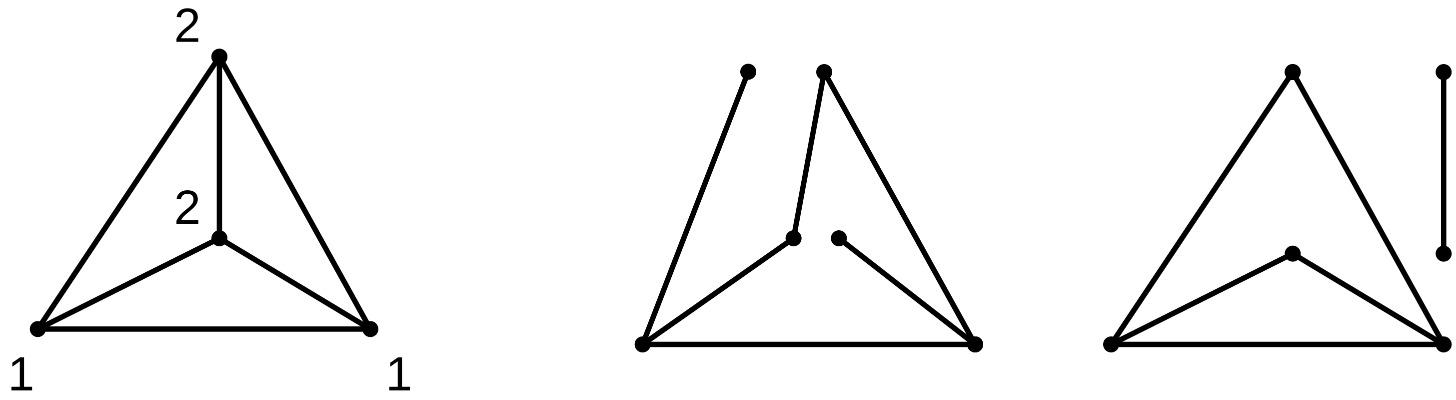
$$e(X) + b(V - X) \geq b(V)$$

Connected Detachment

$G=(V,E)$: Connected Graph $\qquad b: V \to \mathbf{Z}_+$

Detachment

$$G=(V,E) \longrightarrow \hat{G}=(W,E):$$



Split each vertex $v \in V$ into $b(v)$ vertices. Each edge should be incident to some corresponding vertices.

Connected Detachment

Theorem (Nash-Williams [1985])

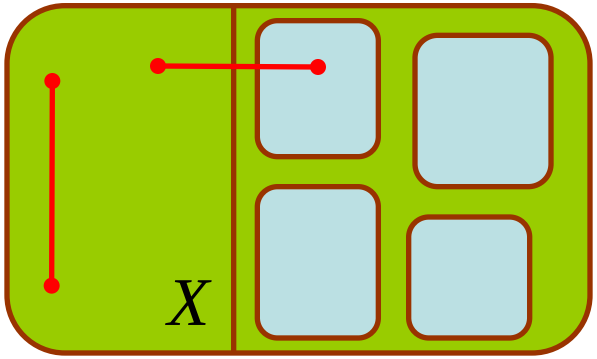
There exists a connected b-detachment of G.

$$\Longleftrightarrow\ b(X) \leq e(X) - c(X) + 1, \quad \forall X \subseteq V.$$

$c(X)$: Number of Connected Components in $G \setminus X$.

$\Rightarrow$ Consider a b-detachment.

Shrink each connected component in $G \setminus X$.



Number of vertices: $b(X) + c(X)$.

Number of edges: $e(X)$.

If the resulting graph is connected,

$$b(X) + c(X) \leq e(X) + 1.$$

Connected Detachment

Original Proof

Matroid Intersection (Nash-Williams [1985])

Alternative Proofs

Matroid Partition (Nash-Williams [1992])

Orientation (Nash-Williams [1995])

Connected Detachment

$f(X) := e(X) - c(X) + 1$ Submodular

$f(V) = |E| + 1, \quad f(\phi) = 0.$

$\Leftarrow$ $b \in P(f)$ $\quad \exists h \in B(f) \cap \mathbf{Z}^V, \quad h \geq b.$

$s \in V$

$$y(v) := \begin{cases} h(v) & (v \neq s) \\ h(s) - 1 & (v = s) \end{cases}$$

$y \in B(e)$

$\exists$ Orientation $\vec{G}$ with in-deg$(v) = y(v), \quad \forall v \in V.$

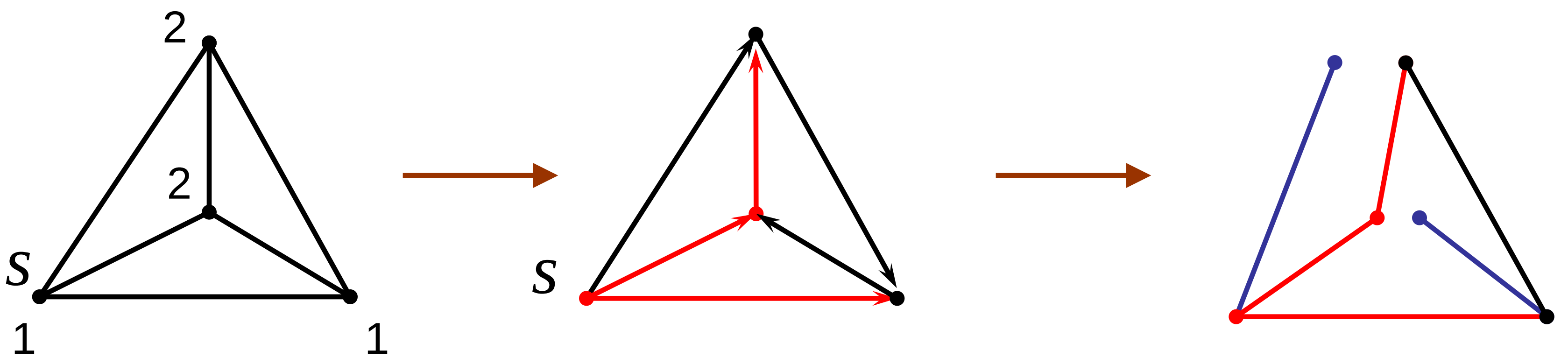
R: Set of vertices reachable from s.

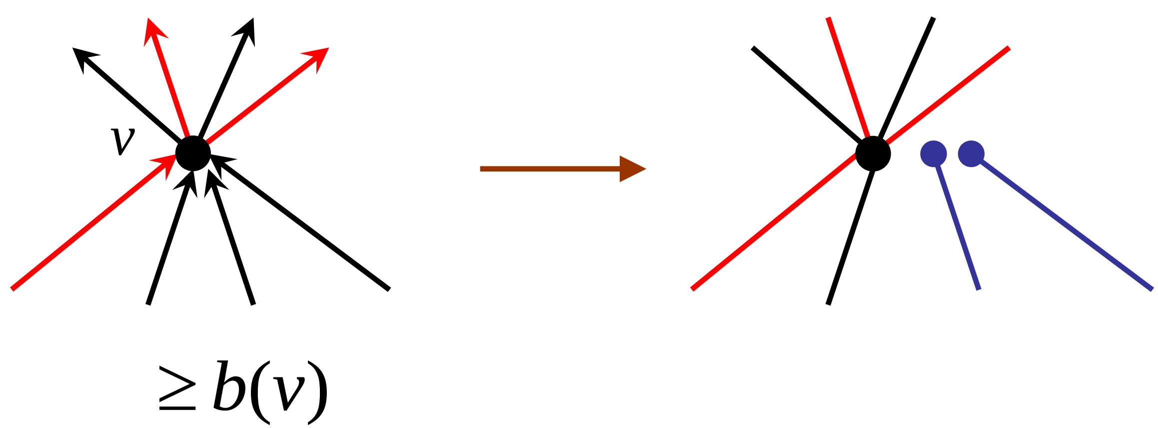
$\underline{e(R)} = y(R) = h(R) - 1 \leq \underline{f(R) - 1}$

$R = V$

Connected Detachment

An orientation connected from a root s such that in-deg$(v) \geq b(v)$ for every $v \neq s$ and in-deg$(s) \geq b(s) - 1$.





Connected Detachment

Testing Feasibility → Submodular Function Minimization

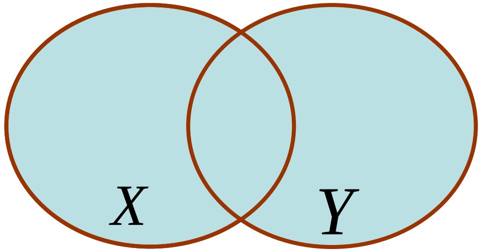
How to Find a Connected Detachment ?

$O(b(V)^{3.5}+m)$ Nagamochi [2006]

Application to Inferring Molecular Structure

$O(nm)$ Iwata & Jordan [2007]

Intersecting Submodular Functions



Intersecting:

$$X \cap Y \neq \phi, \quad X \setminus Y \neq \phi, \quad Y \setminus X \neq \phi.$$

$$f : 2^V \rightarrow \mathbf{R}$$

Intersecting Submodular:

$$f(X) + f(Y) \geq f(X \cap Y) + f(X \cup Y)$$

$\forall X, Y \subseteq V$: Intersecting

Intersecting Submodular Functions

$f : 2^V \rightarrow \mathbf{R}$ Intersecting Submodular $f(\phi) = 0$

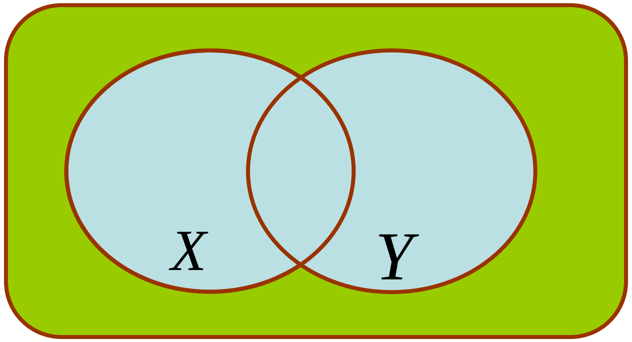
$$P(f) = \{x \mid x \in \mathbf{R}^V, \forall Y \subseteq V, x(Y) \leq f(Y)\}$$

Theorem (Lovász [1977])

There exists a fully submodular function $\tilde{f} : 2^V \rightarrow \mathbf{R}$ such that $P(f) = P(\tilde{f})$.

$$\tilde{f}(X) = \min\{\sum_{i=1}^{k} f(X_i) \mid \{X_1, \ldots, X_k\} : \text{partition of } X\}$$

Crossing Submodular Functions



Crossing:

$$X \cap Y \neq \phi, \quad X \cup Y \neq V,$$

$$X \setminus Y \neq \phi, \quad Y \setminus X \neq \phi.$$

$$f : 2^V \to \mathbf{R}$$

Crossing Submodular:

$$f(X) + f(Y) \geq f(X \cap Y) + f(X \cup Y)$$

$\forall X, Y \subseteq V$: Crossing

Crossing Submodular Functions

$f : 2^V \to \mathbf{R}$ Crossing Submodular $f(\phi) = 0$

$$B(f) = \{x \mid x \in \mathbf{R}^V, x(V) = f(V), \forall Y \subseteq V, x(Y) \le f(Y)\}$$

Theorem (Frank [1982], Fujishige [1984])

There exists a fully submodular function $\tilde{f} : 2^V \to \mathbf{R}$ such that $B(f) = B(\tilde{f})$, provided that $B(f)$ is nonempty.

Bi-truncation Algorithm Frank & Tardos [1988].

Graph Orientation

$G=(V,E)$: Graph $\qquad b: V \to \mathbf{Z}_+$

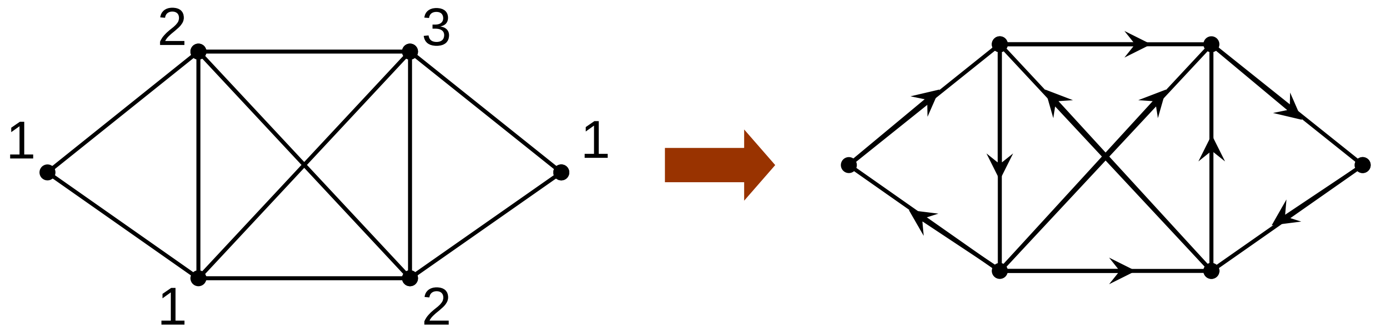
$b \in B(f)$

$e(X)$: Number of Edges Incident to X.

There exists an k-arc-connected orientation $\vec{G}$ with in-deg$(v)=b(v)$ for every $v\in V$.

$\Longleftrightarrow$

$$b(X) \le e(X) - k, \quad \forall X \subseteq V,$$
$$b(V) = e(V).$$



Graph Orientation

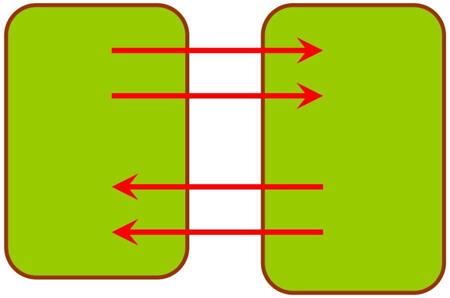
When is $B(f)$ nonempty?

Theorem (Nash-Williams [1960])

There exists an k-arc-connected orientation of G.

$\Updownarrow$

$G:\ 2k$-edge-connected



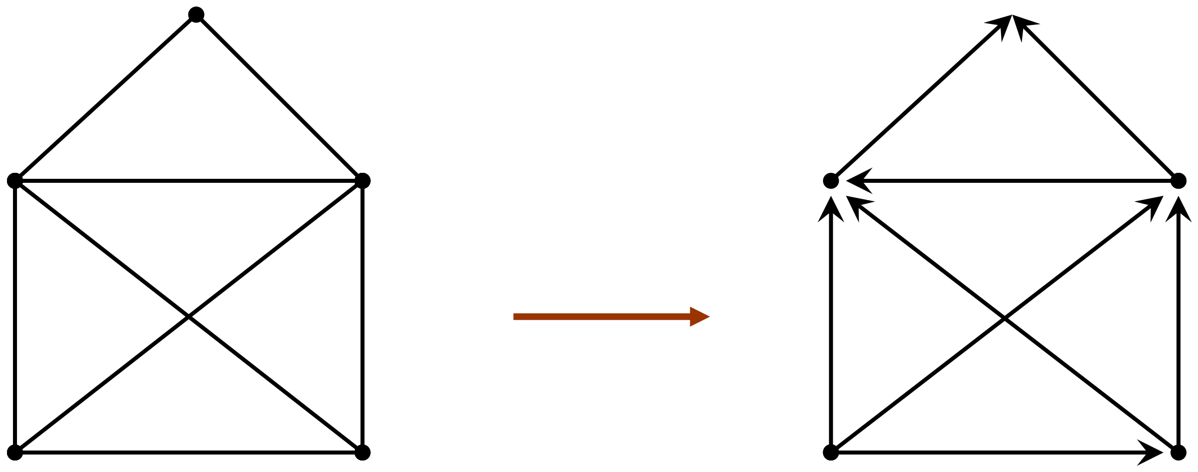
$$x(v) := d(v)/2 \quad (v \in V)$$

$$x \in B(f)$$

Minimax Acyclic Orientation

$G = (V, E)$: Graph

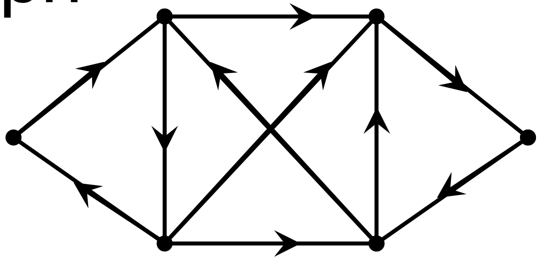
Find an acyclic orientation that minimizes the maximum in-degree

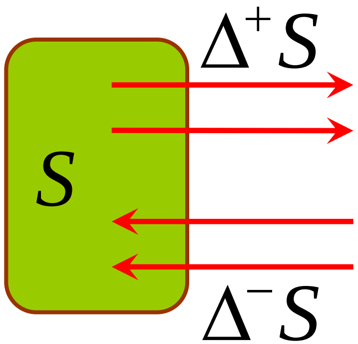


Submodular Flows

Edmonds & Giles (1977)

$G=(V,E)$: Directed Graph





$\underline{c}, \overline{c} : A \to \mathbf{R}$ Capacity

$d : A \to \mathbf{R}$ Cost

$f : 2^V \to \mathbf{R}$ Crossing Submodular Function

$$f(\phi) = f(V) = 0$$

$$\text{Minimize} \quad \sum_{a \in A} d(a)x(a)$$

$$\text{subject to} \quad x(\Delta^+ S) - x(\Delta^- S) \le f(S), \quad \forall S \subseteq V$$

$$\underline{c}(a) \le x(a) \le \overline{c}(a), \quad \forall a \in A.$$

Submodular Flows

- Totally Dual Integral (TDI)

  Edmonds & Giles (1977)

- Polynomial Algorithms Modulo SFMin

  Grötschel, Lovász, Schrijver (1981)

  Frank (1984), Cunningham & Frank (1985)

  Frank & Tardos (1987)

  Fujishige, Röck, Zimmermann (1988)

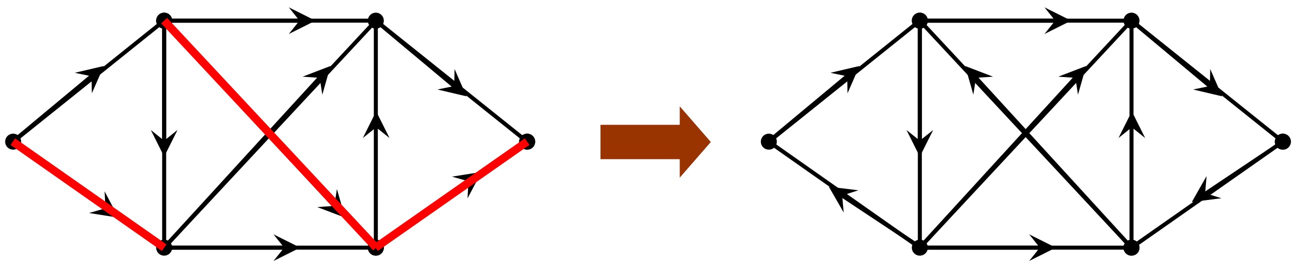
  Iwata (1997),

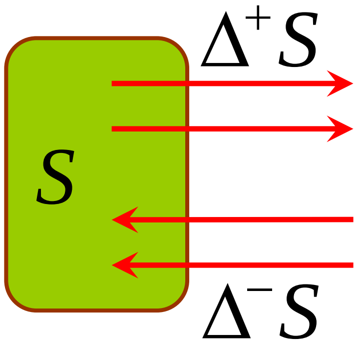
  Iwata, McCormick, Shigeno (2000,2003,2005)

  Fleischer, Iwata, McCormick (2002)

Graph Orientation

$\vec{G} = (V, A)$: Reference Orientation





$d : A \to \mathbf{R}$ Reorientation Cost

$$\begin{aligned}
\text{Minimize} \quad & \sum_{a \in A} d(a) x(a) \\
\text{subject to} \quad & x(\Delta^+ S) - x(\Delta^- S) \le \kappa(S) - k, \quad \forall S \subset V, \\
& 0 \le x(a) \le 1, \qquad \forall a \in A.
\end{aligned}$$