Approximation Algorithms for Submodular Optimization

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Symmetric Submodular Functions

$f: 2^V \to \mathbf{R}$

Symmetric $f(X) = f(V \setminus X), \quad \forall X \subseteq V.$

Symmetric Submodular Function Minimization

$$\min\{ f(X) \mid \phi \neq X \neq V \}?$$

$O(n^3\gamma)$ Queyranne (1998)

Minimum Cut Algorithm by MA-ordering

Nagamochi & Ibaraki (1992)

Minimum Degree Ordering Nagamochi (2007)

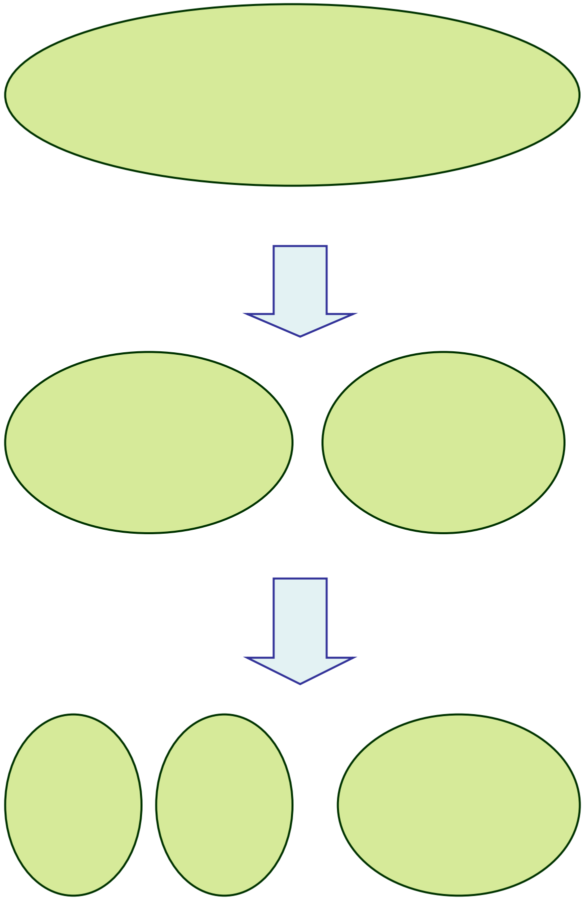
Submodular Partition

Minimize $\sum_{i=1}^{k} f(V_i)$

subject to $V = V_1 \cup \cdots \cup V_k$

$V_i \cap V_j = \phi \ (i \neq j)$

Greedy Split



f : Monotone or Symmetric

$2(1-1/k)$-Approximation

Zhao, Nagamochi, Ibaraki (2005)

Questions

What Kind of Approximation Algorithms
Can Be Extended to Optimization Problems
with Submodular Cost or Constraints ?

Cf. Submodular Flow (Edmonds & Giles, 1977)

How Can We Exploit Discrete Convexity
in Design of Approximation Algorithms?

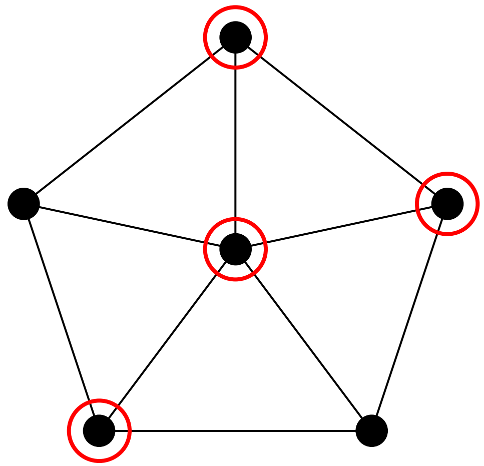
Cf. Ellipsoid Method
(Grötschel, Lovász & Schrijver, 1981)

Submodular Vertex Cover

Graph $G = (V, E)$

Submodular Function

$$f : 2^V \to \mathbf{R}_+$$



Find a Vertex Cover $S \subseteq V$ Minimizing $f(S)$

2-Approximation Algorithm

Goel, Karande, Tripathi, Wang (FOCS 2009)

Iwata & Nagano (FOCS 2009)

Relaxation Problem

Convex Programming Relaxation (CPR)

Minimize $\hat{f}(x)$

subject to $x(u) + x(v) \geq 1 \quad (\forall e = (u, v) \in E)$

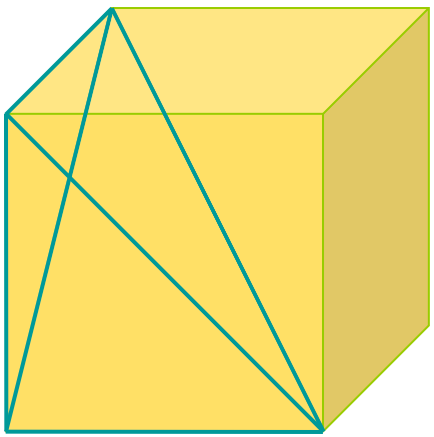
$$x(v) \geq 0 \quad (\forall v \in V)$$

CPR has a half-integral optimal solution.

→ 2-Approximation Algorithm

Proof of Half-Integrality

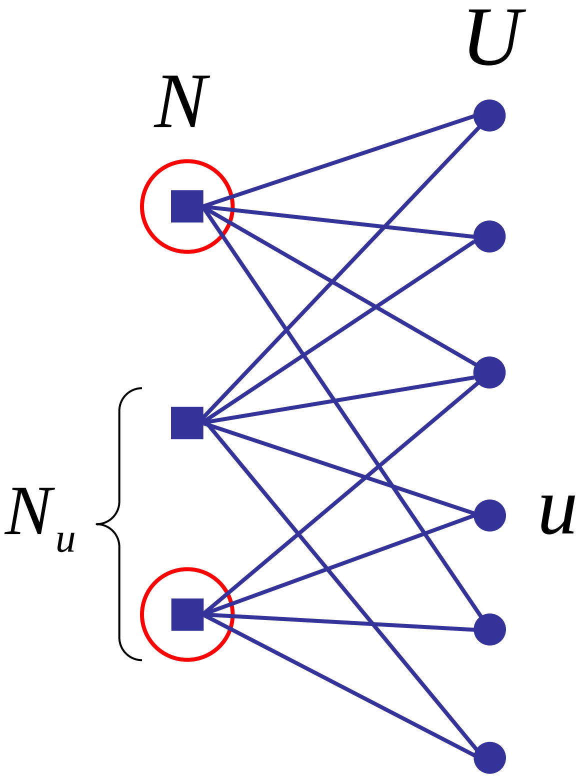
$$\sigma : V \to \{1, \ldots, n\}$$



$$\begin{pmatrix} x_u - x_v \geq 0 \quad (\sigma(u) < \sigma(v)) \\ x_u + x_v \geq 1 \quad (\forall e = (u,v) \in E) \\ x_v \geq 0 \quad (\forall v \in V) \end{pmatrix}$$

Every nonsingular submatrix
in the coefficient matrix has
a half-integral inverse.

Submodular Cost Set Cover



Find $X \subseteq N$ Covering U with Minimum $f(X)$.

$$\eta := \max_{u \in U} |N_u|$$

η-Approximation

Rounding Algorithm

Primal-Dual Algorithm

Koufogiannakis & Young (ICALP 2009)

Greedy η -Approximation Algorigm for Monotone Submodular Functions

Submodular Multiway Partition

Chekuri & Ene (FOCS 2011)

Minimize $$\sum_{i=1}^{k} f(V_i)$$

subject to $$V = V_1 \cup \cdots \cup V_k$$

$$V_i \cap V_j = \phi \ (i \neq j)$$

$$s_i \in V_i \quad (i = 1, \ldots, k)$$

f : Nonnegative Submodular

Relaxation Problem

Convex Programming Relaxation

Minimize $\sum_{i=1}^{k} \hat{f}(x_i)$

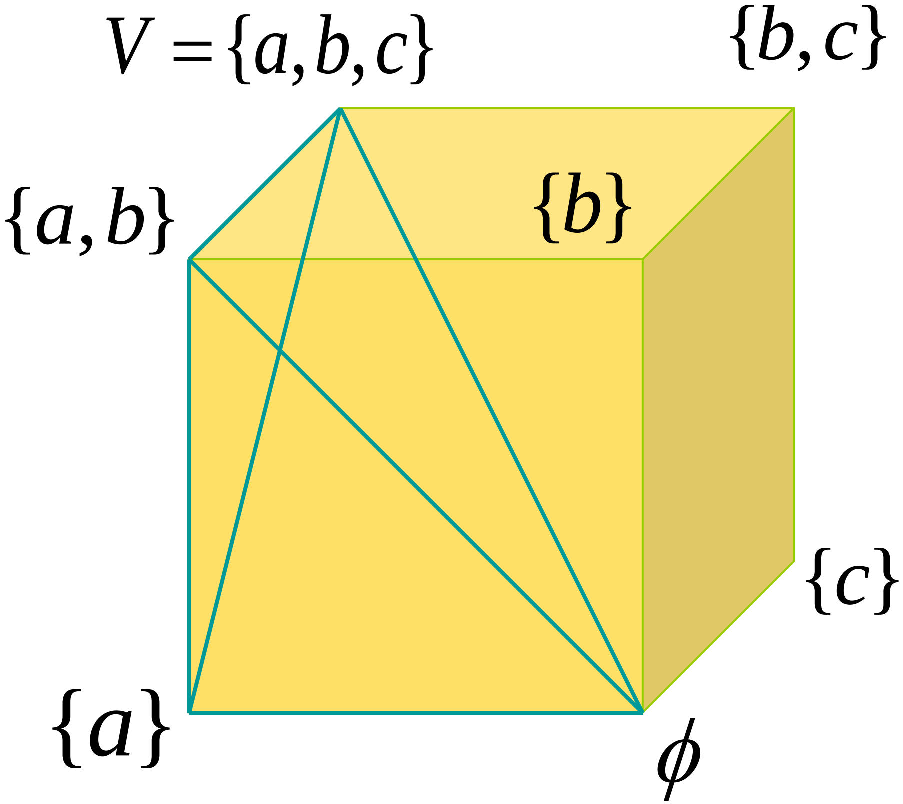
subject to $\sum_{i=1}^{k} x_i(v) = 1 \quad (\forall v \in V)$

$x_i(s_i) = 1 \quad (i = 1, \ldots, k)$

$x_i(v) \geq 0 \quad (i = 1, \ldots, k, \forall v \in V)$

Ellipsoid Method → x_i^* : Optimal Solution

Discrete Convexity



Lovász (1983)

$\hat{f}$: Linear Interpolation

$\hat{f}$: Convex

⇕

f : Submodular

$\theta \in [0,1]$: Chosen Uniformly at Random

$X := \{v \mid x(v) \geq \theta\}, \qquad \hat{f}(x) = \mathrm{E}[f(X)]$

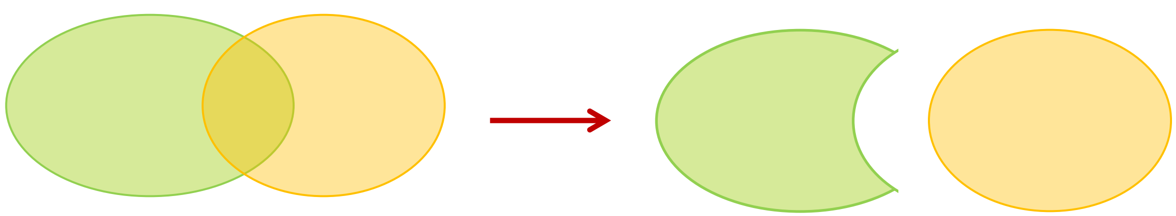
Rounding Scheme

f : Symmetric Submodular

$\theta \in [0,1]$: Chosen Uniformly at Random

$$A_i := \{v \mid x_i(v) \geq \theta\} \quad (i = 1,\ldots,k), \qquad U := V - \bigcup_{i=1}^{k} A_i$$

Uncross $(A_1,\ldots,A_k)$



Return $(A_1,\ldots,A_{k-1},A_k \cup U)$

→ 3/2-Approximate Solution

Rounding Scheme

f : Nonnegative Submodular

$\delta \in (\frac{1}{2}, 1]$: Chosen Uniformly at Random

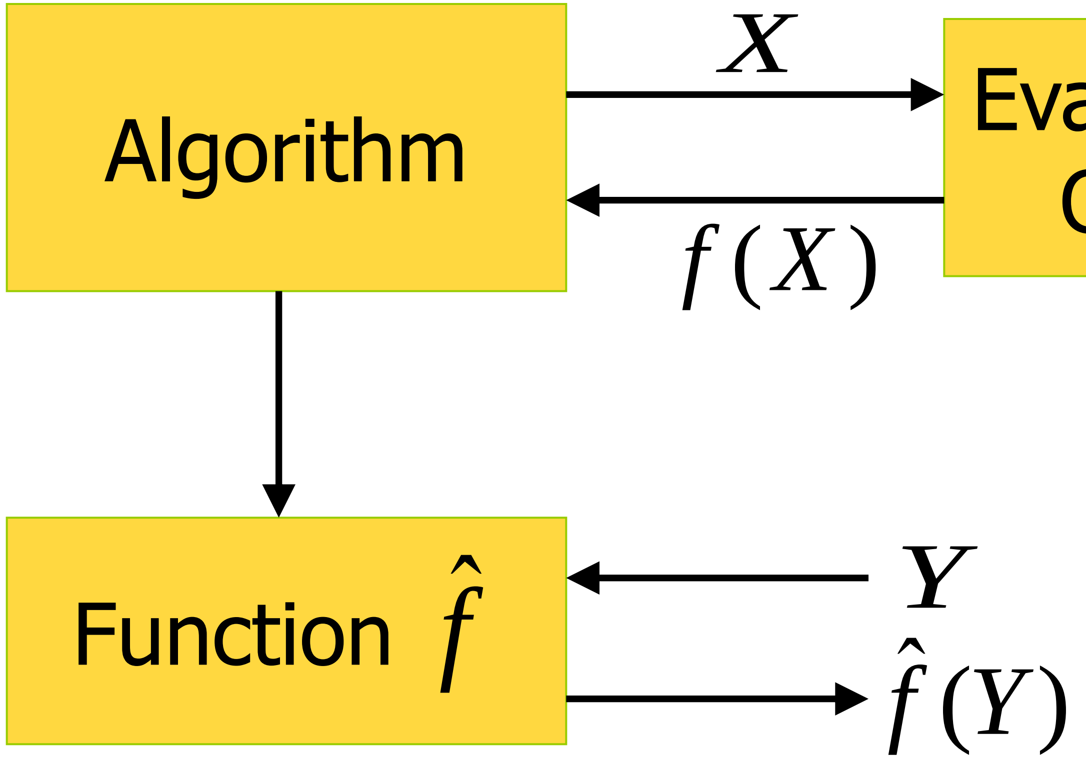
$A_i := \{v \mid x_i(v) \geq \theta\} \quad (i = 1, \ldots, k), \qquad U := V - \bigcup_{i=1}^{k} A_i$

Return $(A_1, \ldots, A_{k-1}, A_k \cup U)$

$\longrightarrow$ 2-Approximate Solution

Improvement over the $(k-1)$-Approximation by Zhao, Nagamochi, & Ibaraki (2005)

Approximating Submodular Functions



Approximating Submodular Functions

Goemans, Harvey, Iwata & Mirrokni (SODA 2009)

$$f(\phi) = 0, \quad f(X) \geq 0, \forall X \subseteq V.$$

Construct a set function $\hat{f}$ such that

$$\hat{f}(X) \leq f(X) \leq \alpha(n) \hat{f}(X), \quad \forall X \subseteq V.$$

For what function α is this possible?

Algorithm with $\alpha(n) = O(\sqrt{n} \log n)$ for monotone submodular functions

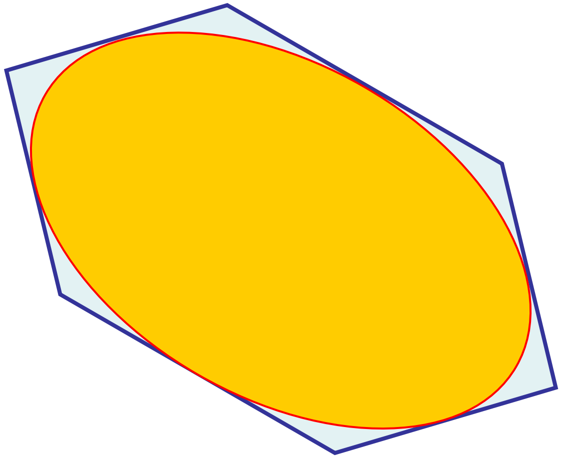
Ellipsoidal Approximation

K : Centrally Symmetric Convex Body

$(x \in K \Rightarrow -x \in K)$

$E(A)$: Maximum Volume Inscribed Ellipsoid

(The John Ellipsoid)



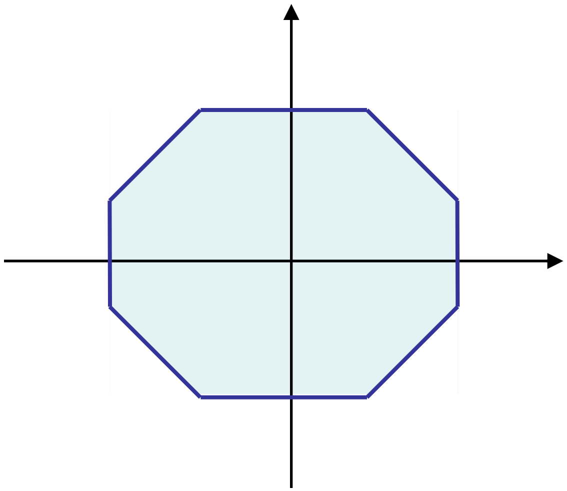
$$E(A) \subseteq K \subseteq \sqrt{n}E(A)$$

Symmetrized Polymatroids

f : Monotone Submodular Function $f(\phi)=0$

$$P(f)=\{x \mid x\in \mathbf{R}_{+}^{V}, \sum_{v\in S} x(v)\le f(S), \forall S\subseteq V\}$$

$$Q(f)=\{x \mid x\in \mathbf{R}^{V}, \sum_{v\in S} |x(v)|\le f(S), \forall S\subseteq V\}$$



John Ellipsoid of $Q(f)$

Axis-Aligned $E(D)$

D : Diagonal

Symmetrized Polymatroids

$E(D)$: John Ellipsoid of $Q(f)$ $\qquad p(u) = \dfrac{1}{D_{uu}} \quad (u \in V)$

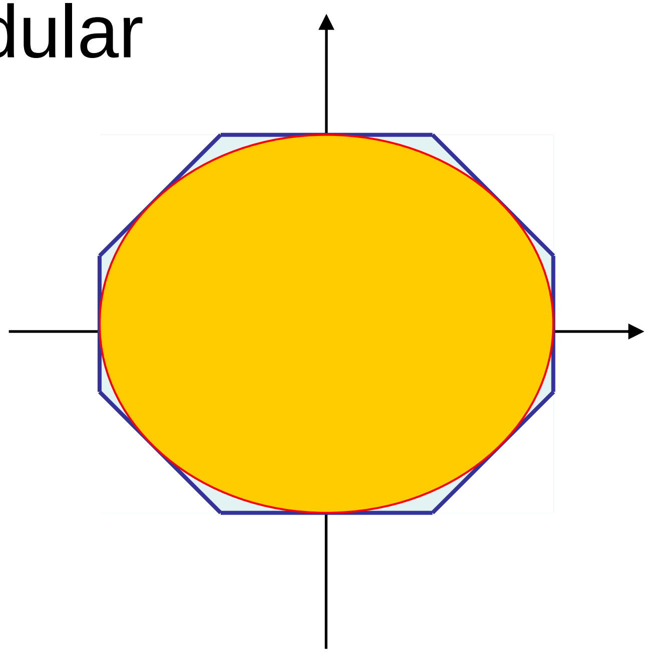
$$\hat{f}(S) := \sqrt{\sum_{u \in S} p(u)} = \max\{\chi_S y \mid y \in E(D)\}$$

$\hat{f}$: Monotone Submodular

$$f(S) = \max\{\chi_S y \mid y \in Q(f)\}$$

$$E(D) \subseteq Q(f) \subseteq \sqrt{n} E(D)$$

$$\hat{f}(S) \le f(S) \le \sqrt{n}\,\hat{f}(S)$$



Submodular Load Balancing

Svitkina & Fleischer (FOCS 2008)

$f_1, \ldots, f_m$: Monotone Submodular Functions

$$\min_{\{V_1, \ldots, V_m\}} \max_j f_j(V_j)?$$

$$f_j(X) := \sum_{v \in X} p_j(v)$$ → Scheduling

2-Approximation Algorithm

Lenstra, Shmoys, Tardos (1990)

$O(\sqrt{n} \log n)$-Approximation Algorithm