Algorithms for Submodular Function Minimization (SFMin)

S. Thomas McCormick

Sauder School of Business, UBC

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Outline

The Greedy Algorithm

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  - E.g., suppose that
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  - Then $3^{\prec_1} = \emptyset$, $3^{\prec_2} = \{1, 2\}$,
    and $2^{\prec_1} = \{1, 3, 4, 5\}$, $2^{\prec_2} = \{1\}$.

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    and $2^{\prec_1} = \{1, 3, 4, 5\}$, $2^{\prec_2} = \{1\}$.
- In this notation we can re-express the main step of Greedy on the ith element in $\prec$ as
  "Make $x_{e_i} \leftarrow f(e_i^{\prec} + e_i) - f(e_i^{\prec})$."

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  - The largest e_i in $S - e_k$ is smaller than k, so induction applies to $S - e_k$ and we get $x(S) - x_{e_k} = x(S - e_k) \leq f(S - e_k)$, or $x(S) \leq f(S - e_k) + x_{e_k} = f(S - e_k) + (f(e_k^{\prec} + e_k) - f(e_k^{\prec}))$.

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  - Thus $x(S) \leq f(S - e_k) + (f(e_k^{\prec} + e_k) - f(e_k^{\prec})) = f(e_{k+1}^{\prec}) + f(S - e_k) - f(e_k^{\prec}) \leq f(S)$.

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- In order to show optimality of the x coming from Greedy, we construct a dual optimal solution.

Dual feasibility

- Here are the dual LPs:

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  - Now $\sum_{S \ni e_k} \pi_S = \sum_{i=k+1}^{n+1} (w_{e_{i-1}} - w_{e_i})$
    $= w_{e_k} - w_{e_{n+1}} = w_{e_k}$, as desired.

Optimality from duality

- For any $x \in B(f)$ and π feasible for the dual, note that

$$\begin{array}{rcl} w^T x & = & \sum_{e \in E} (\sum_{S \ni e} \pi_S) x_e \\ & = & \sum_{S \subseteq E} \pi_S \sum_{e \in S} x_e \\ & = & \sum_{S \subseteq E} \pi_S x(S) \\ & \leq & \sum_{S \subseteq E} \pi_S f(S). \end{array}$$

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  - If $\pi_S = 0$ then both sides are zero.
  - If $\pi_S \neq 0$, then S is $e_k^{\prec}$ for some k.
  - But then $x(S) = \sum_{i<k} x_{e_i} = \sum_{i<k} (f(e_i^{\prec} + e_i) - f(e_i^{\prec})) = f(e_{k-1}^{\prec} + e_{k-1}) - f(\emptyset) = f(e_k^{\prec}) = f(S)$.

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$$\begin{array}{rcl} w^T x & = & \sum_{e \in E} (\sum_{S \ni e} \pi_S) x_e \\ & = & \sum_{S \subseteq E} \pi_S \sum_{e \in S} x_e \\ & = & \sum_{S \subseteq E} \pi_S x(S) \\ & \leq & \sum_{S \subseteq E} \pi_S f(S). \end{array}$$

- Since we already proved that the Greedy output $x \in B(f)$ and our π is feasible, we only need to show that $w^T x = \sum_{S \subseteq E} \pi_S f(S)$.
- Consider the above display. The only place there's an inequality is $\sum_{S \subseteq E} \pi_S x(S) \leq \sum_{S \subseteq E} \pi_S f(S)$.
  - If $\pi_S = 0$ then both sides are zero.
  - If $\pi_S \neq 0$, then S is $e_k^{\prec}$ for some k.
  - But then $x(S) = \sum_{i<k} x_{e_i} = \sum_{i<k} (f(e_i^{\prec} + e_i) - f(e_i^{\prec})) = f(e_{k-1}^{\prec} + e_{k-1}) - f(\emptyset) = f(e_k^{\prec}) = f(S)$.
  - Thus we get equality, and so x is (primal) optimal (and π is dual optimal).

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  - As we saw in the proof, the constraint for $S = e_k^{\prec}$ is tight for each $e_k \in E$.
- Therefore M is the lower triangular matrix:

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- This also shows that $v^{\prec}$ is a vertex, as it follows from M being nonsingular.

Outline

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  - Intuition: as we move k earlier in $\prec$, $v_k^{\prec}$ gets bigger; as we move k later in $\prec$, $v_k^{\prec}$ gets smaller.

Exchange capacities

- We call this step length $\alpha = [f(l^{\prec} + l) - f(l^{\prec})] - [f(l^{\prec} + k + l) - f(l^{\prec} + k)]$ the exchange capacity of the consecutive pair (l, k), and denote it as $c(k, l; v^{\prec})$.

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  - Since $x(E) = f(E)$ is a constraint of $B(f)$, all $x \in B(f)$ have the constant sum $f(E)$. Thus it is not a surprise that $|v_k^{\prec} - v_k^{\prec'}| = |v_l^{\prec} - v_l^{\prec'}| = c(k, l; v^{\prec})$.

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  - Unfortunately it turns out that computing $c(k, l; x)$ is provably as difficult as SFMin.

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  - GLS then extend this to show a strongly polynomial running time.

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- These kinds of "combinatorial" LPs often have 0–1 optimal solutions.
- Even better, we guess (see below) that there exists an optimal solution to the dual where only one π_S is positive, say $\pi_{S^*} = 1$.

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  - Thus $x^*(E) = x^*(S^*) + x^*(E - S^*) = f(S^*) + u(E - S^*)$, proving that S^* induces a dual optimal solution.

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  - This is the form of the LP that we will use.
  - This LP is quite close to the Greedy LP, except that the objective is the piecewise linear $y^-(E)$ instead of $x(E)$, and this makes solving the problem *much* harder.

Outline

SFMin weak duality, complementary slackness

- Here is weak duality for these LPs:

$$
\begin{aligned}
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- If we can achieve this picture along with $y(S) = f(S)$, it *proves* that y and S jointly solve SFMin.

SFMin weak duality, complementary slackness

- Here is weak duality for these LPs:

$$\begin{array}{rcll} y^-(E) & \leq & y^-(S) & \text{tight if } y_e < 0 \implies e \in S \\ & \leq & y(S) & \text{tight if } e \in S \implies y_e \leq 0 \\ & \leq & f(S) & \text{tight if } S \text{ is } y\text{-tight.} \end{array}$$

  Complementary slackness is equivalent to the tightness conditions that ensure that each inequality is an equality.

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  - Then $y = \sum_{i \in \mathcal{I}} \lambda_i v^i$ is a succinct certificate proving that $y \in B(f)$.

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- This can be done with standard linear algebra techniques in $O(n^3)$ time.

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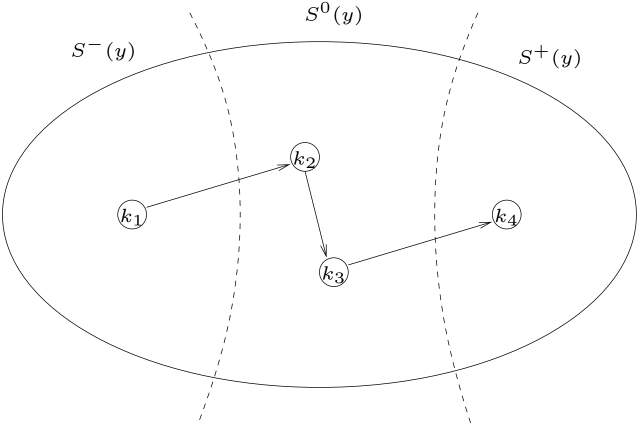
  - But unfortunately computing $c(k, l; y)$ is as hard as SFMin.
  - And if we don't have any $\prec_i$ with (l, k) consecutive in $\prec_i$, then how can we change the representation $y = \sum_{i \in \mathcal{I}} \lambda_i v^i$ to track this $\chi_k - \chi_l$ direction?

SFMin augmenting paths

Assume that we have the situation as in the picture below, where (k_2, k_1) is consecutive in $\prec_1$, (k_3, k_2) is consecutive in $\prec_2$, and (k_4, k_3) is consecutive in $\prec_3$.



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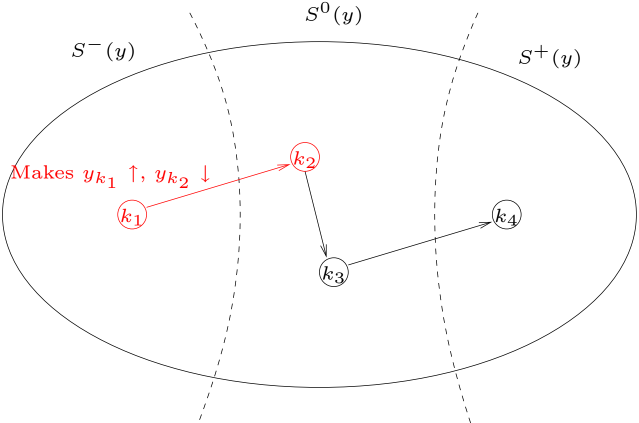
We could swap k_2 and k_1 in $\prec_1$ to $\uparrow y_{k_1}$ and $\downarrow y_{k_2}$, but this wouldn't increase $y^-(E)$.



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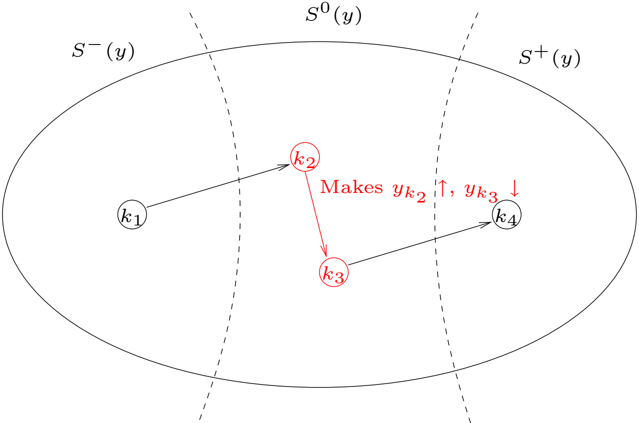

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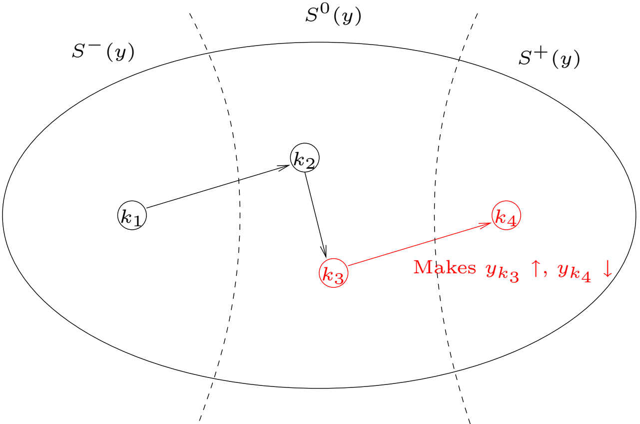

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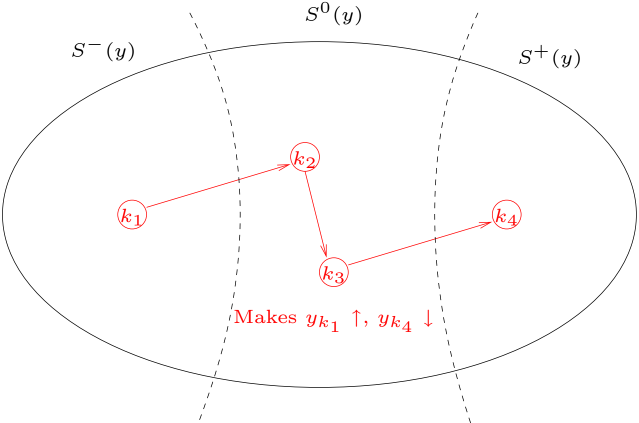

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  - But then we could extend the augmenting path to k along arc $k \to l$ coming from consecutive pair (l, k), contradicting that $l \notin S^*$.

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- The same proof works with a more general definition of arcs: Put $e \to g \in A$ whenever $g \prec_i e$ for some $i \in \mathcal{I}$.

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- I just showed that S^* is at the left of every $\prec_i$.
- Now S^* at the left of $\prec_i$ implies that $v^i(S^*) = f(S^*)$.
- Then $y(S^*) = \sum_{i \in \mathcal{I}} \lambda_i v^i(S^*) = \sum_{i \in \mathcal{I}} \lambda_i f(S^*) = f(S^*) \sum_{i \in \mathcal{I}} \lambda_i = f(S^*)$.
- Thus S^* is y-tight, the third complementary slackness condition, and so S^* is indeed optimal for SFMin.
- This proof is very much in the same spirit as the Max Flow / Min Cut augmenting path proof.
- The same proof works with a more general definition of arcs: Put $e \to g \in A$ whenever $g \prec_i e$ for some $i \in \mathcal{I}$.
- The "only" remaining thing to do is to find some way to arrange augmentations so there is only a polynomial number of them.

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- Augmentation amounts depend on the λ_i, which can be arbitrarily small.
- These are some of the reasons why it took many, many years to figure out how to get a combinatorial SFMin algorithm, and why Cunningham's SFMin algorithm was only pseudo-polynomial.

Current state of the art in SFMin

(Taken from S. T. McCormick (2006). Submodular Function Minimization. Chapter 7 in the *Handbook on Discrete Optimization*, Elsevier, K. Aardal, G. Nemhauser, and R. Weismantel, eds, 321–391.; see my webpage for updated version.)

	Cunningham for General SFM [13], Sec. 3.1	Schrijver [76, 84], Schrijver-PR [22], Sec. 3.2	Iwata, Fleischer, and Fujishige [49, 45], Sec. 3.3	Iwata Hybrid [47], Sec. 3.3.4	Orlin [71], Sec. 3.4.1	Iwata and Orlin [51], Sec. 3.4.2
Pseudo-polyn. running time	$O(Mn^6 \log(Mn) \cdot$ EO)					
Weakly polyn. running time			$O(n^5 \text{EO} \log M)$ [49], Sec. 3.3.1	$O((n^4\text{EO} + n^5) \cdot \log M)$ **(*)**		$O((n^4\text{EO} + n^5) \cdot \log(nM))$
Strongly polyn. running time		$O(n^7\text{EO} + n^8)$ [22, 84]	$O(n^7 \text{EO} \log n)$ [49], Sec. 3.3.2	$O((n^6\text{EO} + n^7) \cdot \log n)$	$O(n^5\text{EO} + n^6)$ **(*)**	$O((n^5\text{EO} + n^6) \log n)$
Fully comb. running time			$O(n^9 \text{EO} \log^2 n)$ [45], Sec. 3.3.3	$O(n^8 \text{EO} \log^2 n)$		$O((n^7\text{EO} + n^8) \log n)$ **(*)**

Outline

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Outline

- Optimizing submodular functions
  - The Greedy Algorithm
  - Edges of $B(f)$
- SFMin algorithms
  - An algorithmic framework
  - Algorithm-izing the dual LPs
- Combinatorial Hull
  - Carathéodory is a bottleneck
  - Avoiding linear algebra
  - Combinatorial hull and membership
  - Algorithmic ideas for combinatorial hull

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- Suppose that we have x, $z \in B(f)$, $y(E) = f(E)$, and $\tilde{x} \leq \tilde{y} \leq \tilde{z}$.
- **Theorem (Fujishige):** Then $y \in B(f)$.

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- This projection of $B(f)$ along one component is a g-polymatroid (and all g-polymatroids arise this way).

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  - Suppose that $y \in B(f)$.
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- Suppose, e.g., that x and z are vertices of $B(f)$ coming from linear orders $\prec_x$ and $\prec_z$.
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  - ...and apply induction to $\tilde{x} \in B^{\#}(\tilde{f})$ to get $\tilde{x}$ as a combinatorial hull from vertices.

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  - It does not appear to be efficient (so far). That is, we don't have a combinatorial hull equivalent to Carathéodory's Theorem.

Outline

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- In these terms we have shown that if $V(f)$ is the set of vertices of $B(f)$, then $\mathrm{combhull}(V(f)) = B(f)$, and $B(f) = \mathrm{combhull}(B(f))$.
- What we have not shown is, starting from $V(f)$, how many iterations of the combinatorial hull operation are necessary to get to an arbitrary point of $B(f)$.

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- Spoiler alert: I'm not going to give you such an algorithm.
- Hopefulness: But I will give you some tools you might use to construct such an algorithm.

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- Can we also do this for combinatorial hull?

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  - NO! A simple counterexample shows that we can have, e.g., S tight for y and z but not tight for x.
  - But if we have that $\tilde{x}_e < \tilde{y}_e < \tilde{z}_e$ for all $e \in \tilde{E}$ with $x_e < z_e$ (i.e., if y is strictly interior wherever possible), then it's fairly easy to show that S tight for y implies that it is also tight for x and z.

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  - Since $y(S)$ is modular, $\hat{f}(S)$ is submodular.
  - Now S proves that $y \notin B(f)$ iff $y(S) > f(S)$ iff $0 > f(S) - y(S) = \hat{f}(S)$, and so $y \in B(f)$ iff $0 \in B(\hat{f})$.

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- Thus a polynomial combinatorial algorithm that uses combinatorial hull to solve membership of 0 in $B(\hat{f})$ is a worthwhile target.

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  - If we had a membership subroutine that didn't use linear algebra, then we'd have a linear algebra-free SFMin algorithm.
- Thus a polynomial combinatorial algorithm that uses combinatorial hull to solve membership of 0 in $B(\hat{f})$ is a worthwhile target.
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- The problem is symmetric between x and z: If we can succeed in constructing a point $z \in B(\hat{f})$ with $\tilde{z} \geq 0$ (or prove that no such z exists), then we could run the same algorithm with signs reversed to get some $x \in B(\hat{f})$ with $\tilde{x} \leq 0$ (or prove that no such x exists).

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  - If $v_n > 0$, then $-y_n = y(E) - y_n = y(E - \{n\}) \leq \hat{f}(E - \{n\}) = v(E - \{n\}) = v(E) - v_n = -v_n$ certifies that $0 \notin B(\hat{f})$.

Outline

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- ... for any terminal subset S of $\prec$ we can assume that $v(S) \leq 0$ (same proof).
- So now let's try to find combinatorial hull moves that will modify v into the $\tilde{x}$ we need.
  - All we need to do is to "re-distribute" the negativity in the terminal elements of v to make every individual component non-positive (not just the terminal partial sums).

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- This movement is along the edge which is the convex hull of v and v' (this sounds bad), but since $v' - v$ has all signs non-positive (after projecting out coordinate n), this is also a valid combinatorial hull operation (which sounds good).

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- In all three case we make real progress.

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- Again we make real progress in all cases.

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- This is the problem with combinatorial hull: Unlike convex hull, you cannot arbitrarily pile on an operation that works in one place (e.g., $v^2 - v$ is a good direction w.r.t. v) and necessarily have it work in another place (e.g., v^2 doesn't have the right signs w.r.t. v').

(Tentative) conclusions

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3. But we don't have an alternative to combinatorial hull in hand either ...