# Do Linear Programs Dream of Oriented Matroids When They Sleep?

Jesús A. De Loera

Partly based on work with subsets of
R. Hemmecke, J. Lee, S. Kafer, L. Sanità,
C. Vinzant, B. Sturmfels, I. Adler, S. Klee, and Z. Zhang

10th Cargese Conference— September 2019
Dedicate to the memory of Frédéric Maffray

# THIS TALK IS ABOUT

## The GEOMETRY of

## LINEAR OPTIMIZATION...

Minimize $\mathbf{c}^T\mathbf{x}$ subject to $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq 0$;

Oriented Matroids part of the history of LP: Rockafellar, Bland, Fukuda, Terlaky, Todd, etc

**Main Message:** Given an LP, we can insert it or embedded as part of a larger **oriented matroid** and win!

**MY GOAL**: Show you 3 examples giving insight for the simplex method and log-barrier interior point methods.

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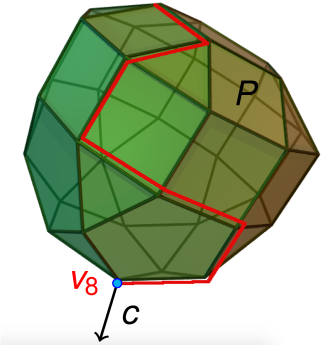
# Outline

1. Oriented Matroids and the Simplex-method

2. Oriented Matroids and Interior-point Methods

# Recall the simplex method...

- The simplex method **walks** along the graph of the polytope, each time moving to a better and better cost vertex!



# BIG ISSUE 1:

# Is there a polynomial bound of the diameter in terms of the number of facets and dimension?

WARNING. If diameter is exponential, then all simplex algorithms will be exponential in the worst case.

$(facets(P) - dim(P)) + 1 \leq \text{Diameter} \leq (facets(P) - dim(P))^{\log(dim(P))}.$

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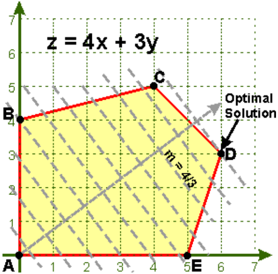
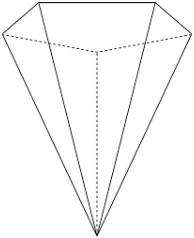
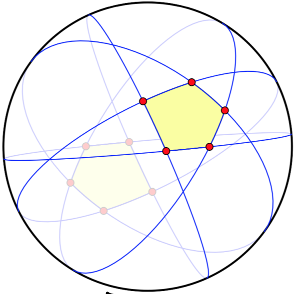
# From polytopes to oriented matroids

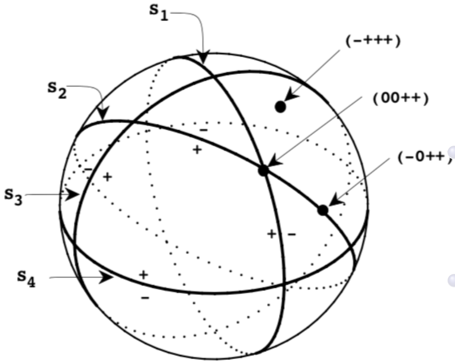




# FROM ARRANGEMENTS TO ORIENTED MATROIDS

Consider a hyperplane arrangement of $n$ hyperplanes in $\mathbb{R}^r$, intersect it with sphere $S^{r-1}$.




- These **sign vectors** constitute an abstraction of hyperplane arrangements, an ORIENTED MATROID!
- Covectors of minimal support are called *cocircuits* of OM (Vertices!)
- We can also call the 1-skeleton the cocircuit graph.
- Covectors of maximal support are called *topes* of OM. (polytopal regions!)
- The collection of sign vectors representing cells are *covectors*.

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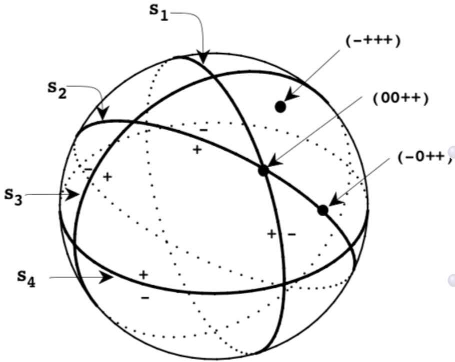


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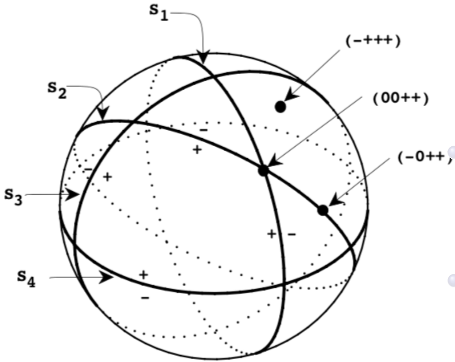


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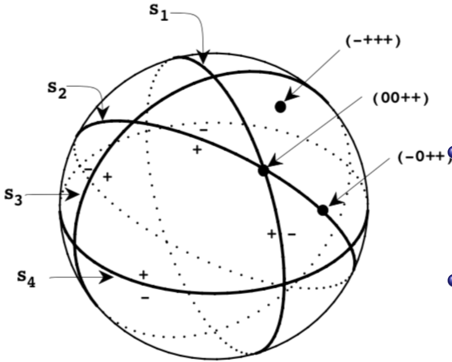


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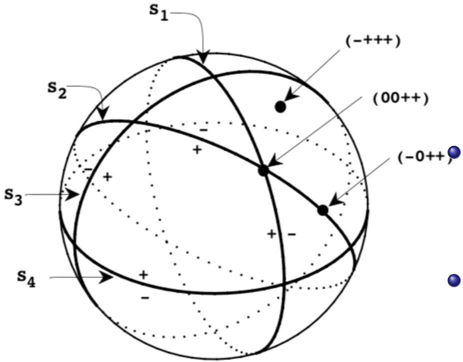


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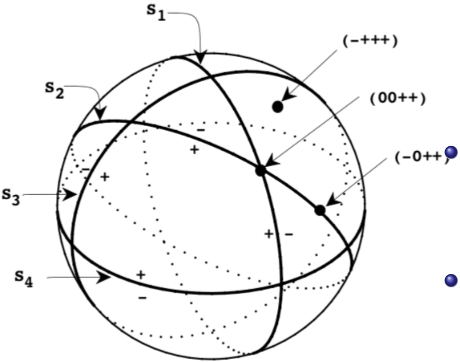


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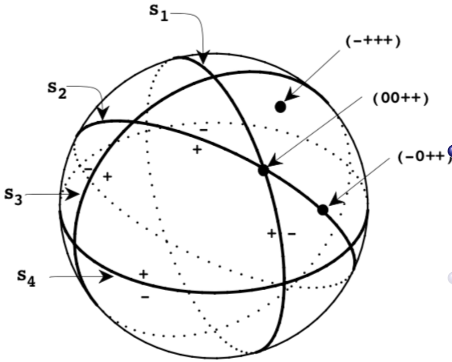


- Want to bound the distance between any two cocircuits in the graph of an oriented matroid.
- The diameter of an Oriented Matroid is the diameter of the **cocircuit graph.**
- Denote by $\Delta(n, r)$ the largest diameter on Oriented Matroids with cardinality $n$ and rank $r$.

**KEY QUESTION**

How do we bound $\Delta(n, r)$?

This is of course related to the Hirsch conjecture for polytopes!!

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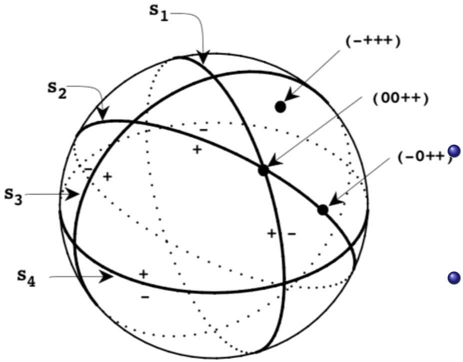


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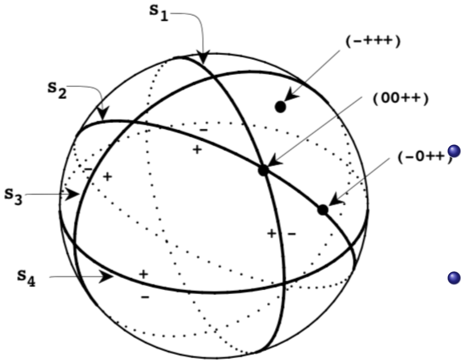


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# Conjectures

**Conjecture 1**

For all $n$ and $r$,

$$\Delta(n, r) = n - r + 2.$$

Given a sign vector $X$, the antipodal $-X$ has all signs reversed (that is, for all $e \in E$, $(-X)_e = -X_e$).

**Lemma**

*Antipodals are at distance at least $n - r + 2$. Thus diameter is at least $n - r + 2$.*

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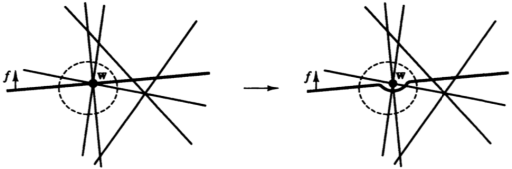
# Simplification Lemmas

Definition: A rank $r$ oriented matroid is **uniform** , when every cocircuit $X$ is defined by $r - 1$.

**Lemma (Adler-JDL-Klee-Zhang)**

*For all $n, r$, $\Delta(n, r)$ is achieved by some* **uniform** *oriented matroid of cardinality $n$ and rank $r$.*




**Conjecture 2**

Only the distance of antipodals can achieve the diameter length. That is, for $X, Y \in \mathcal{C}^*$, $X \neq -Y$, $d(X, Y) \leq n - r + 1$.

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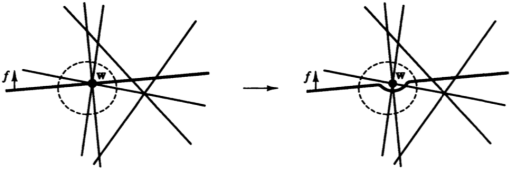


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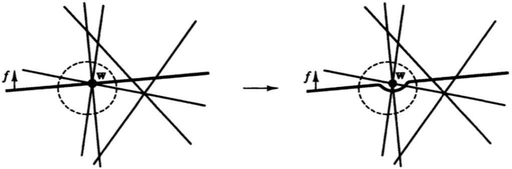


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# Low rank or corank and small $n, r$

## Theorem (Adler-JDL-Klee-Zhang.)

*$\Delta_r(n, r) = n - r + 2$ When*

- *For $r \leq 3$ and for $n - r \leq 3$.*

- *A counterexample needs to have at least 10 elements! (Classification of small oriented matroids)*

| | | $n = 4$ | $n = 5$ | $n = 6$ | $n = 7$ | $n = 8$ | $n = 9$ | $n = 10$ |
|---|---|---|---|---|---|---|---|---|
| $r = 3$ | | *1* | *1* | *4* | *11* | *135* | *4382* | *312356* |
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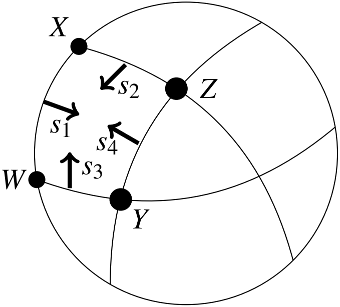
# Proof idea for rank 3




$\ell(P_W) + \ell(P_Z) \leq 4 + 2(n-4) = 2n - 4$. So $d_M(X, Y) \leq n - 2$.

# An Quadratic Diameter Upper Bound

**Theorem (Adler-JDL-Klee-Zhang.)**

*The diameter of all rank r oriented matroids with n elements satisfies*

$$\Delta(n,r) \leq \max\left\{ \lceil \frac{\min(r-1, n-r+1)}{2} \rceil (n-r+1),\ n-r+2 \right\}.$$

**Which means the diameter is quadratic!!**

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# Proof $\Delta(n,r) \leq (r-1)(n-r+2)$

- The proof is by induction on the rank $r$ of the oriented matroid.
- For $r = 2$. A rank two oriented matroid is a circle divided by $2n$ points (these are $n$ 0-spheres).




- The graph is a $2n$-gon, diameter equals $1 \cdot n = (r-1)(n-r+2)$.
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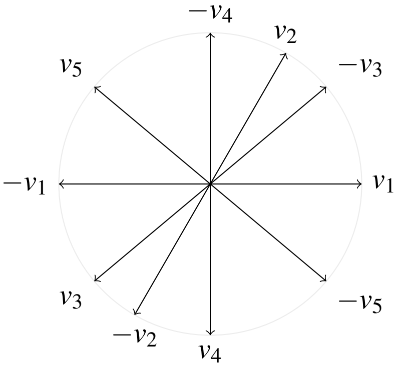


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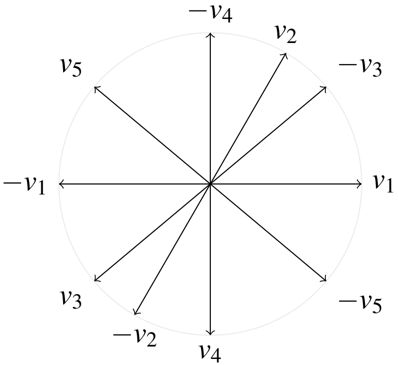


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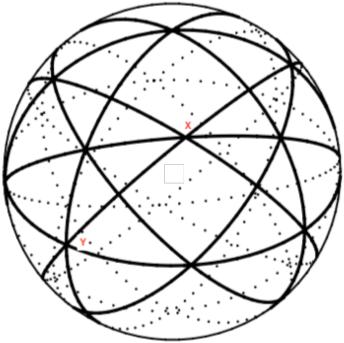
- CASE 1: If $X, Y$ are both contain in the same pseudo-sphere $S_i$,

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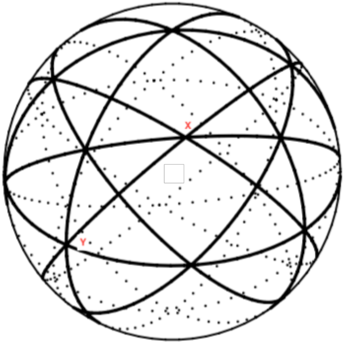


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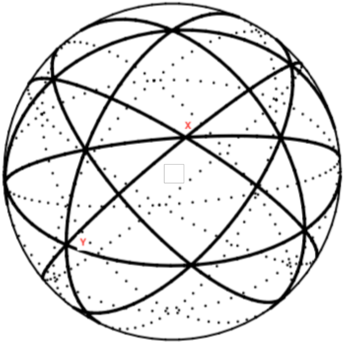


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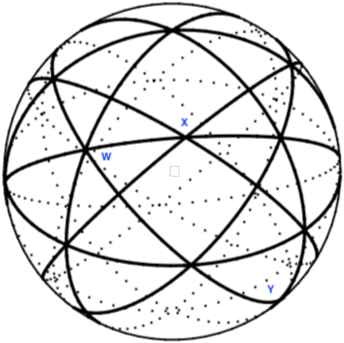
- The cocircuit $Y$ is the intersection of $r-1$ spheres. Take $r-2$ of those, consider the restriction. What is it?
- It is an oriented matroid of rank 2, an arrangement of $n-r-2$ many 0-spheres (points) along a circle $\gamma$.
- The circle $\gamma$ intersects all other spheres, at least one contains $X$. Call that cocircuit $W$.



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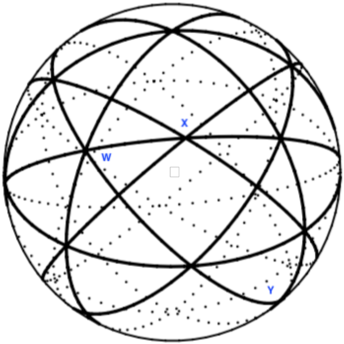

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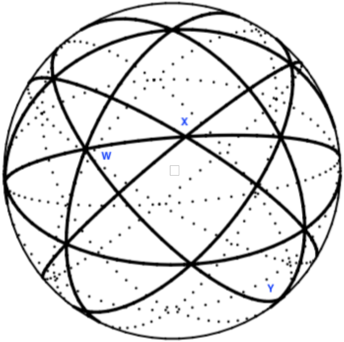

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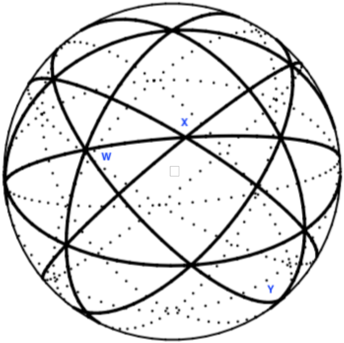

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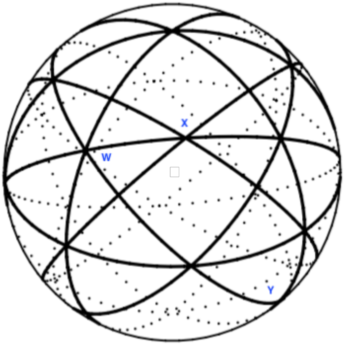

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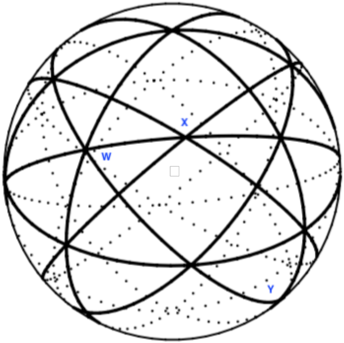

- The distance $d(X, Y)$ is no more than $d(Y, W)$ plus $d(W, X)$.
- We apply induction twice:

$$d(Y, W) \leq 1 \cdot (n - r - 2) \leq n - r + 2$$

$$d(W, X) \leq (r - 2)((n - 1) - (r - 1) + 2) = (r - 2)(n - r + 2)$$

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# Hirsch Conjecture and Oriented Matroids

F. Santos constructed a 20-polytope with 40 facets, with diameter 21. It violates (polytope) Hirsch conjecture! We can construct an Oriented Matroid containing Santos's counterexample as a tope.

**Lemma**

*There exists an Oriented Matroid with cardinality 40 and rank 21 that violates Conjecture 2.*

**Conjecture**

For all cocircuits $X, Y \in \mathcal{C}^*(\mathcal{M})$ in the same tope $T$, there exists a path $P$ such that

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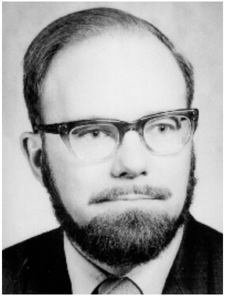
# BIG ISSUE 2: Fast pivot rules??

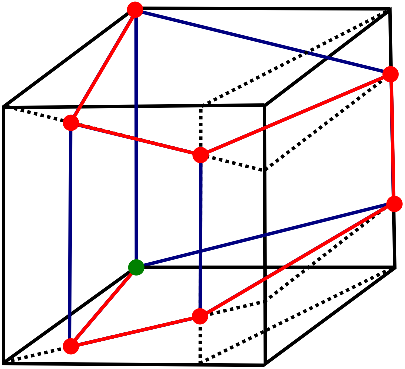
Is there a pivoting rule that turns the simplex algorithm

into a polynomial time algorithm for solving linear programming problems?

# PIVOT RULES BEHAVE BADLY!!



First bad example Klee-Minty cubes 1972



TODAY 2019: For most pivot rules we have exponential examples!!

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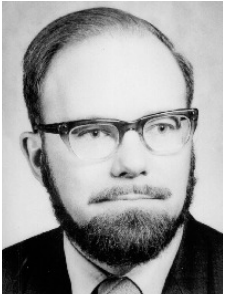

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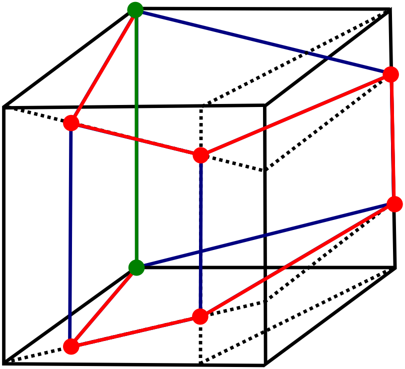

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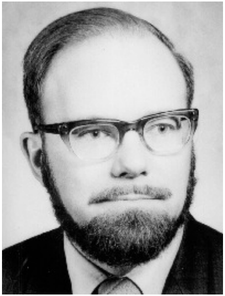

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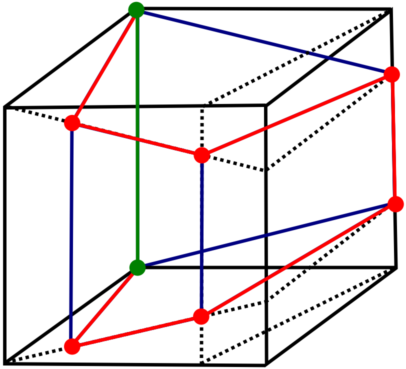

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# Possible edges are Minimal Linear Dependences!

- Let $A$ be matrix that defines our LP $min\{cx : Ax = b, 0 \leq x \leq u\}$

$$A = \left[\ x^1 \ \middle|\ x^2 \ \middle|\ \cdots \ \middle|\ x^n \ \right]$$

  Consider the finite sets of all minimal linear dependent subsets of columns

  $\mathcal{C}(A) := \{A_S :\ A_S$ has *linearly* dependent columns;
  $A_{S-e}$ has *linearly* independent columns$\}$.

- These are the CIRCUITS of the matrix $A$, denoted $\mathcal{C}(A)$.
  - $E = \{1, 2, 3,\ 4,\ 5,\ 6\}$.

$$A = \left[\begin{array}{c|c|c|c|c|c} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$$

  - Example: Is $\{2, 3, 4, 6\} \in \mathcal{C}(A)$?
  - A1: Yes. $A_2 + A_3 + A_4 - A_6 = 0$, yet
    $\det[A_2|A_3|A_4] = 1$, $\det[A_2|A_3|A_6] = 1$,
    $\det[A_2|A_4|A_6] = -1$, $\det[A_3|A_4|A_6] = 1$.

# ALL CIRCUITS ARE THE NEW LEGAL MOVES!

- $$A = \begin{bmatrix} 2 & 1 & 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \quad \mathbf{c} = (1, 1, -1, 0, 0, 0).$$

  What are the circuits of the LP?

  $a_1 = (1, 0, 0, -2, -1, 0)$, $a_2 = (0, 1, 0, -1, -2, 0)$, $a_3 = (0, 0, 1, 0, 0, -1)$, $a_4 = (1, -2, 0, 0, 3, 0)$, $a_5 = (2, -1, 0, -3, 0, 0)$. and their negatives!

- Circuits satisfy the axioms of a MATROID!

(C1) $\emptyset \notin \mathcal{C}$.

(C2) $X, Y \in \mathcal{C},\ X \subset Y \ \Rightarrow\ X = Y$.

(C3) $X, Y \in \mathcal{C},\ X \neq Y,\ e \in X \cap Y \ \Rightarrow$
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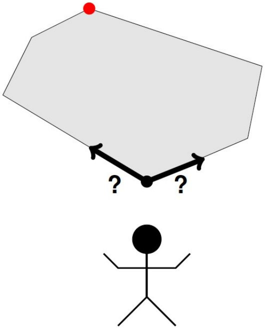
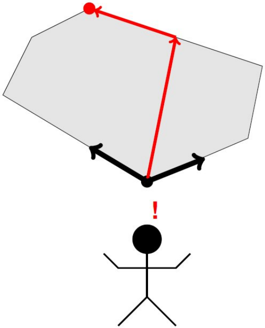
# ENDING THE "EDGES ONLY" PIVOTING POLICY:

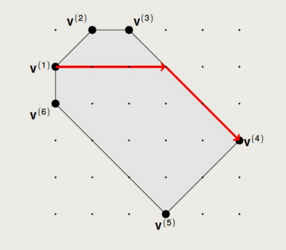
We wish to solve

$$\min\{\, \mathbf{c}^{\mathsf{T}}\mathbf{x} \;:\; A\mathbf{x} = \mathbf{b},\; \mathbf{0} \le \mathbf{x} \le u,\; \mathbf{x} \in \mathbb{R}^n \,\}.$$



# BUT ONE CAN ALSO GO THROUGH THE INTERIOR TOO!!




## IDEA: USE *all circuits* OF THE MATRIX $A$ TO IMPROVE

circuits = support minimal elements of $\ker(A)$. We may walk *through the interior* of the polyhedron!

## LEMMA:

Circuits $\mathcal{C}(A)$ contain all possible **edge directions** of ALL polytopes in the family $\{\, \mathbf{z} : A\mathbf{z} = \mathbf{b}, \mathbf{z} \geq \mathbf{0} \,\}$

# EDMONDS-KARP MAX-FLOW ALGORITHM (1972)

A maximum flow algorithm in a network: The number of augmentations in networks with $|E|$ edges and $|V|$ vertices is only $|E| \cdot |V|$ when augmentation directions are always chosen to have the fewest number of arcs, and the augmentation is maximal

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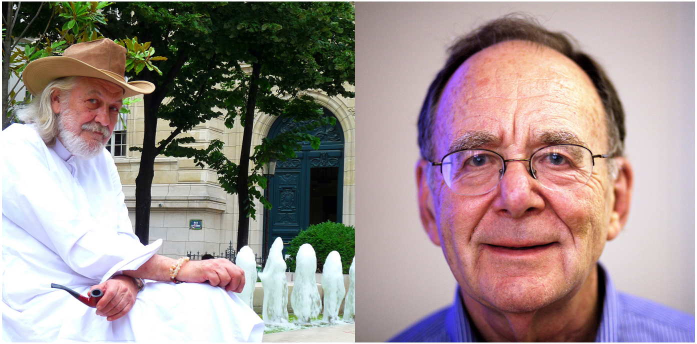

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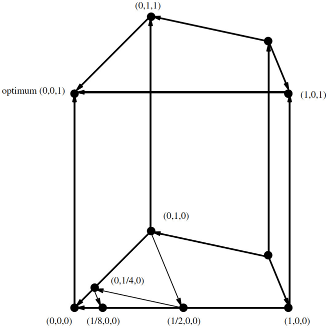
careless augmentation process (using only $a_4, a_5$) does not terminate, zig-zags!!




**Lemma** One reaches an optimal vertex in finitely many steps if golden rule is followed:

*Use an improving circuit, then, while possible, use circuits that add zeros to the solution, once there are none left, we are at a vertex.*

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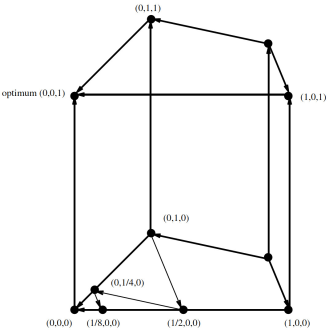


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# HOW MANY CIRCUIT STEPS TO REACH THE OPTIMUM?

Depends on the Pivoting or Augmentation rule!

For a feasible solution $\mathbf{x}_k$, and $\mathcal{T}(A)$ set of improving circuits

**DEFINITION (GREATEST IMPROVEMENT PIVOT RULE)**

Choose $\mathbf{z}$ such that $-\alpha \mathbf{c}^\intercal \mathbf{z}$ is maximized among all $\mathbf{z} \in \mathcal{T}(A)$ and $\alpha > 0$ such that $\mathbf{x}_{k+1} := \mathbf{x}_k + \alpha \mathbf{z}$ is feasible.

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**Definition (Greatest Improvement pivot rule)**

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# Theorem (JDL, R. Hemmecke, and J. Lee

**Theorem**

*Let $A \in \mathbb{Z}^{d\times n}$, $\mathbf{b} \in \mathbb{Z}^d$, $u \in \mathbb{Z}^n$, $\mathbf{c} \in \mathbb{Z}^n$, define the LP*

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*Let $\mathbf{x}_0$ be an initial feasible solution, let $\mathbf{x}_{\min}$ be an optimal solution, and let $\delta$ denote the greatest absolute value of a determinant among all $d \times d$ submatrices (i.e., bases) of $A$.*

- *The number of* **greatest-improvement** *augmentations needed to reach an optimal solution of the LP is no more than* $2n\log(\delta\, \mathbf{c}^\intercal(\mathbf{x}_0 - \mathbf{x}_{\min})) + n$.

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## Lemma 1: Sign-compatible representation

Every $\mathbf{z} \in \ker(A) \cap \mathbb{R}^n$ has a sign-compatible representation using circuits $g \in \mathcal{C}(A)$.

$$\mathbf{z} = \sum_{i=1}^{n} \lambda_i \mathbf{g}_i, \quad \lambda_i \in \mathbb{R}_+.$$

## Lemma 2: Geometric decrease of objective fct.

Let $\epsilon > 0$ be given. Let $\mathbf{c}$ be an integer cost vector.
Let $\mathbf{x}_{\min}$ and $\mathbf{x}_{\max}$ be a minimizer and maximizer of the LP problem and $\mathbf{x}_k$ at the $k$-th iteration of an algorithm.
Let $f^{\min} := \mathbf{c}^\intercal \mathbf{x}_{\min}, f^{\max} := \mathbf{c}^\intercal \mathbf{x}_{\max}$, and $f^k = \mathbf{c}^\intercal \mathbf{x}_k$ the objective-function values.
Suppose that the algorithm guarantees that for the $k$-th iteration:

$$(f^k - f^{k+1}) \geq \beta(f^k - f^{\min})$$

Then we reach a solution with $f^k - f^{\min} < \epsilon$ in no more than $2 \log\left((f^{\max} - f^{\min})/\epsilon\right)/\beta$ augmentations.

# Proof of Theorem

1. Observe that

$$0 > \mathbf{c}^\top(\mathbf{x}_{\min} - \mathbf{x}_k) = \mathbf{c}^\top \sum \alpha_i \mathbf{g}_i = \sum \alpha_i \mathbf{c}^\top \mathbf{g}_i \geq -n\Delta,$$

   where $\Delta > 0$ is the largest value of $-\alpha\mathbf{c}^\top\mathbf{z}$ over all $\mathbf{z} \in \text{Circuits}(A)$ and $\alpha > 0$ for which $\mathbf{x}_k + \alpha\mathbf{z}$ is feasible.

2. Rewriting this, we obtain

$$\Delta \geq \frac{\mathbf{c}^\top(\mathbf{x}_k - \mathbf{x}_{\min})}{n}.$$

3. Let $\alpha\mathbf{z}$ be the greatest-descent augmentation applied to $\mathbf{x}_k$, leading to $\mathbf{x}_{k+1} := \mathbf{x}_k + \alpha\mathbf{z}$. Then we see that $\Delta = -\alpha\mathbf{c}^\top\mathbf{z}$ and

$$\mathbf{c}^\top(\mathbf{x}_k - \mathbf{x}_{k+1}) = -\alpha\mathbf{c}^\top\mathbf{z} = \Delta \geq \frac{\mathbf{c}^\top(\mathbf{x}_k - \mathbf{x}_{\min})}{n}.$$

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4 Take $\delta$ the greatest absolute value of a determinant among all $d \times d$ submatrices (i.e., bases) of $A$.

Applying Lemma 2 with $\beta = 1/n$ and $\epsilon = 1/\delta$ then yields a solution $\bar{\mathbf{x}}$ with $\mathbf{c}^\intercal(\bar{\mathbf{x}} - \mathbf{x}_{\min}) < 1/\delta$, obtained within $2n\,\log(\delta\,\mathbf{c}^\intercal(\mathbf{x}_0 - \mathbf{x}_{\min}))$ augmentations.

5 A greatest descent augmentation makes progress in objective value less than $\epsilon/n$, we have

$$\mathbf{c}^\intercal(\mathbf{x}_k - \mathbf{x}_{\min}) = -\sum \alpha_i \mathbf{c}^\intercal \mathbf{g}_i < n \cdot \epsilon/n = \epsilon.$$

6 any vertex with an objective value of at most $\mathbf{c}^\intercal\bar{\mathbf{x}}$ must be optimal. Hence any feasible solution with an objective value of at most $\mathbf{c}^\intercal\bar{\mathbf{x}}$ must be optimal.

7 An optimal basic solution can be found from $\bar{\mathbf{x}}$ in at most $n$ additional augmentations (using again greatest descent but on a sequence of face-restricted LPs).

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6 any vertex with an objective value of at most $\mathbf{c}^\intercal\bar{\mathbf{x}}$ must be optimal. Hence any feasible solution with an objective value of at most $\mathbf{c}^\intercal\bar{\mathbf{x}}$ must be optimal.

7 An optimal basic solution can be found from $\bar{\mathbf{x}}$ in at most $n$ additional augmentations (using again greatest descent but on a sequence of face-restricted LPs).

4 Take $\delta$ the greatest absolute value of a determinant among all $d \times d$ submatrices (i.e., bases) of $A$.

Applying Lemma 2 with $\beta = 1/n$ and $\epsilon = 1/\delta$ then yields a solution $\bar{\mathbf{x}}$ with $\mathbf{c}^\intercal(\bar{\mathbf{x}} - \mathbf{x}_{\min}) < 1/\delta$, obtained within $2n \log(\delta\, \mathbf{c}^\intercal(\mathbf{x}_0 - \mathbf{x}_{\min}))$ augmentations.

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# Hardness of Pivot rules

*How hard is to solve these three pivot rule optimization problems?*
The set of all circuits is finite but can be exponentially large!

- **Theorem**[JDL-Kafer-Sanità] Greatest-improvement and Dantzig pivot rules are NP-hard. But steepest descent can be computed in polynomial time!
- Key idea for hardness: computing a circuit using Greatest-improvement pivot rule and the Dantzig pivot rule is hard to solve for the *fractional matching polytope*.
- **fractional matching polytope** is the (half-integral) polytope given by the standard LP-relaxation for the matching problem. There is a circuit characterization!
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**Gracias!**

**Merci!**

**Danke!**
**Thank you!**