Decreasingly Minimal Orientations and Flows

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Joint work with

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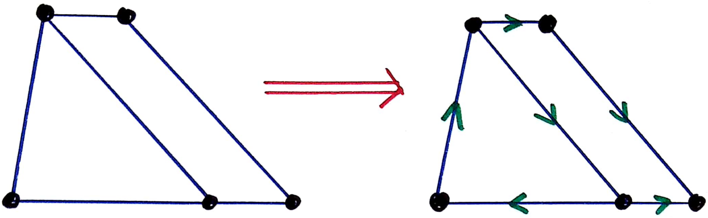
Reports on ARXIV

- A. Frank and K. Murota, *Discrete Decreasing Minimization, Part I:*
  Base-polyhedra with Applications in Network Optimization
  https://arxiv.org/pdf/1808.07600.pdf
- A. Frank and K. Murota, *Discrete Decreasing Minimization, Part II:*
  Views from discrete convex analysis
  https://arxiv.org/pdf/1808.08477.pdf
- A. Frank and K. Murota, *Discrete Decreasing Minimization, Part III:*
  Network flows
  https://arxiv.org/pdf/1907.02673.pdf

Graph orientations



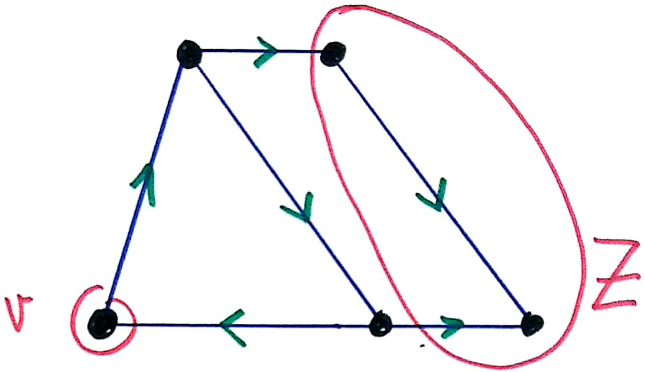
Orienting an undirected edge uv $(= vu)$:

replace uv with a directed edge $(=$ arc$)$ uv or vu

Orienting an undirected graph $G = (V, E)$:

orient each edge of G

In-degree ϱ of a node v and a subset Z



$\varrho(v) = 1$

$\varrho(Z) = 2$

In-degree specified orientation

Theorem (Orientation Lemma, Hakimi, 1965)

Given an in-degree specification $m : V \to \mathbf{Z}$,

$G = (V, E)$ *has an orientation with* $\varrho(v) = m(v)$ *for* $\forall\ v \in V \iff$

$\widetilde{m}(V) = |E|$ *and* $\widetilde{m}(Z) \geq i_G(Z)$ *whenever* $Z \subset V$

($\iff$ $\widetilde{m}(V) = |E|$ *and* $\widetilde{m}(Z) \leq e_G(Z)$ *whenever* $Z \subset V$).

$\widetilde{m}(Z) := \sum[m(v) : v \in Z]$

$i_G(Z)$: number of edges induced by Z

$e_G(Z)$: number of edges with ≥ 1 end-node in Z

In-degree bounded orientation

$f : V \to \mathbf{Z}$: lower bound

$g : V \to \mathbf{Z}$: upper bound $(f \leq g)$

Theorem (F. + Gyárfás, 1976)

$G = (V, E)$ *has an orientation for which*

(A) $\varrho(v) \geq f(v)$ *for* $\forall$ *node* v $\iff$ $\widetilde{f}(Z) \leq e_G(Z)$ *whenever* $Z \subseteq V$

(B) $\varrho(v) \leq g(v)$ *for* $\forall$ *node* v $\iff$ $\widetilde{g}(Z) \geq i_G(Z)$ *whenever* $Z \subseteq V$

(AB) linking property $f(v) \leq \varrho(v) \leq g(v)$ *for* $\forall$ *node* v $\iff$
$\exists$ *an orientation with* $\varrho(v) \geq f(v)$ *and* $\exists$ *an orientation with* $\varrho(v) \leq g(v)$.

(equivalent to earlier results on degree-bounded subgraphs of a bipartite graph)

Theorem (F. + Gyárfás, 1976)

A 2-edge-conn. graph $G = (V, E)$ has a strong orientation for which

(A) $\varrho(v) \geq f(v)$ *for* $\forall$ *node* v $\iff$ $\widetilde{f}(Z) \leq e_G(Z) - c(Z)$ *whenever* $Z \subseteq V$

(B) $\varrho(v) \leq g(v)$ *for* $\forall$ *node* v $\iff$ $\widetilde{g}(Z) \geq i_G(Z) + c(Z)$ *whenever* $Z \subseteq V$

(AB) linking property $f(v) \leq \varrho(v) \leq g(v)$ *for* $\forall$ *node* v $\iff$
$\exists$ *a strong orientation with* $\varrho(v) \geq f(v)$ *and*
$\exists$ *a strong orientation with* $\varrho(v) \leq g(v)$.

($c(Z)$: number of components of $G - Z$)

Corollary

If G has a strong orientation with $\varrho(v) \leq \beta$ for $\forall\, v \in V$, and
G has a strong orientation with $\varrho(v) \geq \alpha$ for $\forall\, v \in V$, then
G has a strong orientation with $\alpha \leq \varrho(v) \leq \beta$ for $\forall\, v \in V$.

In-degree distributions

find an (in-degree bounded) orientation of G in which

the in-degree sequence (or vector) is, intuitively

fair, equitable, egalitarian, as close to uniform as possible, ...

a constant vector $(5, 5, \dots, 5)$ is the most fair

the near-uniform $(5, 5, 4, 4, 4)$ is more 'fair' than $(7, 6, 4, 3, 2)$

capture mathematically the intuitive feeling for 'most fair'

there are several (non-equivalent) definitions:

Possible formal fairness concepts

- the largest component of the vector is as small as possible
- given k, the sum of the k largest components is as small as possible
- the largest component is as small as possible, and subject to this, the number of largest components is minimum

symmetrically:

- the smallest component is as large as possible
- given k, the sum of the k smallest components is as large as possible
- the smallest component is as large as possible, and subject to this, the number of smallest components is minimum

More global ‘fairness’ concepts

the previous fairness definitions are sensitive only for the extreme components of the vector. More global approaches:

- the total deviation $\sum_s |x(s) - m(s)|$ from a specified vector m is minimum (e.g. find a strong orientation with minimum in-degree deviation from m)
- the square-sum $\sum_s x(s)^2$ of the components is minimum
- the difference-sum $\Delta(x) := \sum[|x(s) - x(t)| : s, t \in S]$ is minimum
- decreasingly minimal (dec-min):
  the largest component is as small as possible, within this,
  the second largest component is as small as possible, etc
- increasingly maximal (inc-max):
  the smallest component is as large as possible, within this,
  the second smallest component is as large as possible, etc

Dec-min

reorder decreasingly the components of vector x to obtain $x_\downarrow$
$x = (2, 5, 5, 1, 4) \Rightarrow x_\downarrow := (5, 5, 4, 2, 1)$

x and y value-equivalent: $x_\downarrow = y_\downarrow$

$x <_{\mathrm{dec}} y$ (x is decreasingly smaller than y): if
$x_\downarrow$ is lexicographically smaller than $y_\downarrow$

for a set B of vectors, $x \in B$ is decreasingly minimal (dec-min) if
$x \leq_{\mathrm{dec}} y$ for every $y \in B$

obvious: the dec-min elements are value-equivalent

Egalitarian orientation

Borradaile, Iglesias, Migler, Ochoa, Wilfong, Zhang: BIMOWZ

Egalitarian graph orientation

J. of Graph Algorithms and Applications (2017)

egalitarian orientation: the in-degree sequence is dec-min

motivated by a practical problem in telecommunication

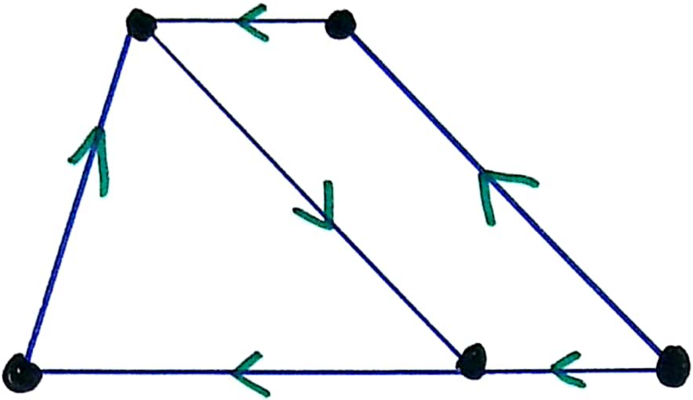
apparently not a perfect name:
an increasingly maximal orientation may also be felt 'egalitarian'

but . . . ? ? ?

Examples

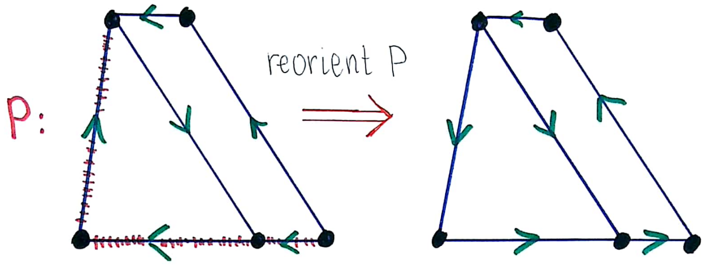
example for an egalitarian orientation: every in-degree is ℓ or $\ell - 1$.

example for a non-egalitarian orientation:



Improving a non-egalitarian orientation

$\varrho(t) = 2$



$\varrho(s) = 0$

non-egalitarian egalitarian

Improving an orientation

local improvement: reorient an *st*-dipath when $\varrho(t) \geq \varrho(s) + 2$

Theorem (BIMOWZ, 2017)

An orientation of G is egalitarian $\Longleftrightarrow$ *there is no local improvement.*

$\Rightarrow$ dec-min and inc-max orientations are the same

(thus the original name 'egalitarian' is legitimate)

questions :

- dec-min in-degree bounded and/or strongly connected orientation (motivated by optimal routing tables of networks)
- are dec-min and inc-max the same for strong orientations, too?

Dec-min strongly connected orientation

BIMOWZ conjectured:

a strong orientation of G is decreasingly minimal $\iff$
$\nexists$ local improvement

local improvement in a strong orientation:

when $\varrho(t) \geq \varrho(s) + 2$ and $\exists$ 2 edge-disjoint *st*-dipaths,
reorient an *st*-dipath [resulting in a strong orientation with dec-smaller in-degree vector]

Theorem (2018+)

A strong orientation of G is dec-min $\iff$ $\nexists$ local improvement.

$\Rightarrow$ dec-min and inc-max are the same for strong orientations, too

. . . but this is not so outright natural since . . .

Strong orientation for mixed graphs

example shows for strong orientations of mixed graphs that

dec-min orientation is NOT the same as
inc-max orientation

the path reversing technique does not suffice
to find a dec-min strong orientation of a mixed graph

before proving the original BIMOWZ conjecture for undirected graphs

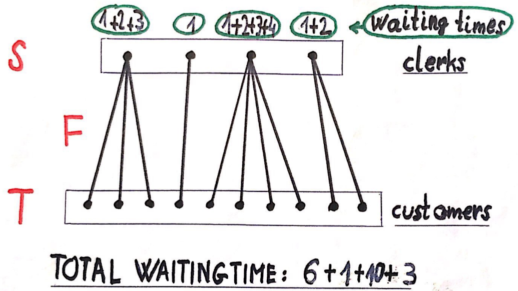
consider a related problem:

Resource allocation: semi-matchings I

$G = (S, T; E)$: bipartite graph

$F \subseteq E$: **semi-matching** when $d_F(t) = 1$ for $t \in T$

Harvey-Ladner-Lovász-Tamir (2006): algorithm to find such an F minimizing the 'total waiting time' $\sum[d_F(s)(d_F(s) - 1) : s \in S]$



Resource allocation: semi-matchings II

$\sum[d_F(s)(d_F(s)-1) : s \in S] = \sum[d_F(s^2) : s \in S] - |S|$ implies:

minimizing total waiting time $=$
minimizing degree square-sum over S

??? min-max theorem for

$\min\{\sum[d_F(s^2) : s \in S] : F \subseteq E$ a semi-matching of $G\}$???

Harada-Ono-Sadakane-Yamashita (2007): algorithm for finding a cheapest semi-matching with min total waiting time

2019+: polyhedral description of semi-matchings with min total waiting time

Resource allocation: extended semi-matchings

Bokal + Brešar + Jerebic (2012): extension to m_T-semi-matching ($d_F(t) = m_T(t)$ for $t \in T$)

Theorem

An m_T-semi-matching F minimizes the total waiting time $\iff$ its degree-vector $(d_F(s) : s \in S)$ on S is decreasingly minimal.

new extension:

Resource allocation: degree-bounded matchings

$G = (S, T; E)$: bigraph, γ: positive integer
$f : (S \cup T) \rightarrow \mathbf{Z}_+$: lower bound, $g : (S \cup T) \rightarrow \mathbf{Z}_+$: upper bound ($f \leq g$)

find a subgraph $F \subseteq E$ of G meeting

$f(v) \leq d_F(v) \leq g(v)$ for $\forall$ $v \in S \cup T$, and $|F| = \gamma$ such that

the degree-vector $(d_F(s) : s \in S)$ on S (!!!) is decreasingly minimal

2018+: algorithm to compute a dec-min F

2019+: algorithm to compute a min-cost dec-min F

based on the known fact: the set of degree-vectors on S of degree-constrained subgraphs of G with γ edges is an M-convex set

Base-polyhedra and M-convex sets

S: ground-set
b: integer-valued submodular function on S

$B = B(b)$: base-polyhedron defined by

$B = \{x \in \mathbf{R}^S : \widetilde{x}(S) = b(S),\ \widetilde{x}(Z) \leq b(Z)$ for $\forall\ Z \subset S\}$

($B(b) \neq \emptyset$, but the empty set is also considered a base-polyhedron, $B(b)$ uniquely determines b)

can also be defined by a supermodular function p:

$B = B'(p) = \{x \in \mathbf{R}^S : \widetilde{x}(S) = p(S),\ \widetilde{x}(Z) \geq p(Z)$ for $\forall\ Z \subset S\}$

($p(X) := b(S) - b(S - X)$: the **complementary** function of b)

$\ddot{B}$: set of integral elements of base-polyhedron B

called an M-convex set in Discrete convex analysis

Operations on base-polyhedra and M-convex sets

the following are base-polyhedra :

- the convex hull of the bases of a matroid $M = (S, r)$ $(= B(r))$
- the translation of $B(b)$ with a vector
  (**matroidal:** if $b = r$ is a matroid rank-function)
- the intersection of $B(b)$ with a box $\{x \in \mathbf{R}^S : f \leq x \leq g\}$
  (the linking property holds)
- a face of $B(b)$
- the sum $B := B(b_1) + B(b_2) + \cdots + B(b_q)$ of base-polyhedra
  (every integral $z \in B$ can be expressed as $z = z_1 + \cdots + z_q$ with integral $z_i \in B(b_i)$)
- $B'(p)$ when p is only crossing supermodular

the corresponding statements hold for M-convex sets

Decreasingly minimal elements of $\ddot{\ddot{B}}$

an element $m \in \ddot{\ddot{B}}$ is **decreasingly minimal** (dec-min) in $\ddot{\ddot{B}}$ if

the largest component of m is as small as possible,
within this, the next largest component of m is as small as possible,
and so on

[increasingly maximal (inc-max) elements are defined analogously]

locally improving $m \in \ddot{\ddot{B}}$:

when $m(t) \geq m(s) + 2$ and $m' := m - \chi_t + \chi_s$ is in $\ddot{\ddot{B}}$
(that is, $\nexists$ m-tight $t\overline{s}$-set)

decrease $m(t)$ by 1 and increase $m(s)$ by 1 (:replace m by m')

Local improving in an M-convex set

implicitly in Groenevelt (1991) and Tamir (1995):

Theorem (2018+)

For an element m of an M-convex set $\ddot{\ddot{B}}$, the following are equivalent.
(A) $\nexists$ *local improving for* m

(B1) m *is dec-min in* $\ddot{\ddot{B}}$
(B2) m *is inc-max in* $\ddot{\ddot{B}}$

$$p(X) := \begin{cases} i_G(X) + 1 & \text{if } \emptyset \subset X \subset V \\ i_G(X) & \text{if } X = \emptyset \text{ or } X = V \end{cases}$$

p is crossing supermodular $\Rightarrow$ $B := B'(p)$ is a base-polyhedron

m is an in-degree vector of a strong orientation $\Longleftrightarrow$ $m \in \ddot{\ddot{B}}$

$\Rightarrow$ BIMOWZ conjecture

Orientations covering a set-function

$h \geq 0$: crossing supermodular

digraph D **covers** h: $\varrho_D(Z) \geq h(Z) \quad \forall \quad \emptyset \subset Z \subset V$

Theorem (A.F. 1980)

$G = (V, E)$ *has an orientation covering* h $\iff$

$$e_{\mathcal{P}} \geq \sum_{i=1}^{q} h(V_i) \quad \text{and} \quad e_{\mathcal{P}} \geq \sum_{i=1}^{q} h(V - V_i)$$

for $\forall$ *partition* $\mathcal{P} = \{V_1, \dots, V_q\}$ *of* V. ($e_{\mathcal{P}}$: $\sharp$ *of edges connecting distinct* V_i*'s)*

for $p := h + i_G$ (crossing supermodular) and $B := B'(p)$ (base-polyhedron)

easy observation: the set of in-degree vectors of orientations of G covering h is the M-convex set $\ddot{B}$.

$\Rightarrow$ dec-min orientation covering h $=$ inc-max orientation covering h

(not true (!) when h is only crossing supermodular and its non-negativity is dropped)

Special cases

$f : V \to \mathbf{Z}$: lower bound
$g : V \to \mathbf{Z}$: upper bound $(f \leq g)$

Theorem (2018+)

A k-edge-con. and in-degree bounded orientation of G is dec-min
$\iff$ $\nexists$ *nodes s, t with*

$$\varrho(t) \geq \varrho(s) + 2, \;\; \varrho(t) > f(t), \;\; \varrho(s) < g(s)$$

for which $\exists$ $k+1$ edge-disjoint st-dipaths.

extends to in-degree bounded and (k, ℓ)-edge-connected orientation

(a digraph is (k, ℓ)**-edge-connected** $(0 \leq \ell \leq k)$ if ℓ-edge-connected and $\exists$ k edge-disjoint dipaths from a root-node to $\forall$ other node)

Characterizing decreasing minimality

$B = B'(p)$: base-polyhedron
$m \in B$: integral element

$Z \subseteq S$ is m-**tight** if $\widetilde{m}(Z) = p(Z)$

X m-**top-set** : $m(t) \geq m(s)$ whenever $t \in X$ and $s \in S - X$

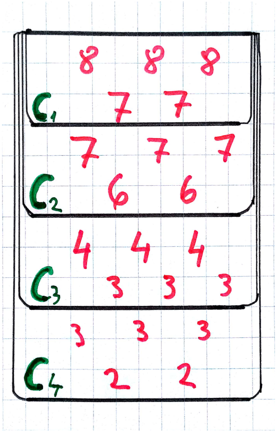
Theorem (2018+)

For $m \in \overset{....}{B}$, *the following are equivalent.*

(A) $\nexists$ *local improving for* m *(*$=$ m *is dec-min)*

(C) $\exists$ *'certificate' chain* $\mathcal{C}$ *of m-tight and m-top sets*
$\emptyset \subset C_1 \subset C_2 \subset \cdots \subset C_\ell$ $(= S)$ *such that*
each difference set $C_i - C_{i-1}$ *is near-uniform.*

Chain certifying decreasing minimality



$$m = (8, 8, 8, 7, 7, 7, 7, 7, 6, 6, \ldots, 2, 2) \in \ddot{\ddot{B}}$$

each C_i is m-top and m-tight (: $\widetilde{m}(C_i) = p(C_i)$)

Canonical certificate chain

for any dec-min element m of $\overset{....}{B}$, define iteratively for $i = 1, 2, \ldots, q$

$\beta_i := \max\{m(s) : s \in S - C_{i-1}\}$

$C_i :=$ smallest m-tight set containing each $s \in S$ with $m(s) \geq \beta_i$

Theorem (2018-19+)

Both the value-sequence $\beta_1 > \beta_2 > \cdots > \beta_q$ and the chain $\mathcal{C} = \{C_1 \subset C_2 \subset \cdots \subset C_q\}$ are independent of the choice of m.

$\Rightarrow$ the 'canonical' chain $\mathcal{C}$ is a certificate for ALL dec-min elements

Algorithmic aspects

2018+: strongly polynomial algorithm for finding a dec-min element m of $\overset{....}{B}$ and the canonical chain $\mathcal{C}$

when $B'(p)$ is small (that is, the values of p can be bounded by a polynomial of $|S|$), the sequence of local improvements provides a polynomial algorithm

in the general case,
the Newton-Dinkelbach algorithm is needed to maximize $\lceil \frac{p(X)}{|X|} \rceil$
along with a subroutine to maximize a supermodular function

Describing the set of all dec-min elements

Theorem (2018+)

Given an integral base-polyhedron B,
$\exists$ a small box T and a face F of B such that

an element $m \in \overset{....}{B}$ is dec-min $\Longleftrightarrow$
m is an integral member of the base-polyhedron $F \cap T$.

$T = \{x \in \mathbf{R}^S : f \leq x \leq g\}$ is **small** if $g(s) - f(s) \leq 1$ for $\forall\ s \in S$

Theorem (2018+)

The dec-min elements of an M-convex set form a matroidal M-convex set.

2018+: strongly polynomial algorithm to compute a min-cost dec-min element of $\overset{....}{B}$

Dec-min optimization on matroids

Edmonds + Fulkerson: given matroids $M_1, M_2, \dots, M_k$ on S, find a basis from each M_i which are disjoint

generalization: find a basis B_i from each M_i such that the vector

$$\sum_{i=1}^{k} \chi_{B_i}$$

is decreasingly minimal (χ_{B_i} is the characteristic vector of B_i)

$B = B(b)$: base-polyhedron defined by the submodular function $b := r_1 + r_2 + \cdots + r_k$

$\Rightarrow$ find a dec-min element of $\ddot{\ddot{B}}$

the special case $M_1 = M_2 = \cdots = M_k$ was solved by Levin and Onn (2016)

Square-sum minimization, I

Fujishige (1980) solved: find an element x of a base-polyhedron B minimizing the square-sum $w(x) := \sum[x(s)^2 : s \in S]$

(there is a unique solution)

discrete version: find an element m of an M-convex set $\ddot{B}$ minimizing the square-sum $w(m)$

different orders:

$$(2,3,3,1) <_{\rm dec} (3,3,3,0) <_{\rm dec} (2,2,4,1) <_{\rm dec} (3,2,4,0)$$

$$w = 23 \quad < \quad w = 27 \quad > \quad w = 25 \quad < \quad w = 29$$

and yet . . .

Square-sum minimization, II

Theorem (2018+)

A member m of an M-convex set $\ddddot{B}$ minimizes the square-sum $w(m)$ over the elements of $\ddddot{B}$ if and only if m is a dec-min member of $\ddddot{B}$.

Theorem (2018+)

$$\min \ \{\sum[m(s)^2 : s \in S] : m \in \ddddot{B}\} \ =$$

$$\max \ \{\hat{p}(\pi) - \sum_{s \in S} \lfloor \frac{\pi(s)}{2} \rfloor \lceil \frac{\pi(s)}{2} \rceil : \pi \in \mathbf{Z}^S\}.$$

($\hat{p}$ is the linear (or Lovász-) extension of p)

the 'easy' inequality max $\leq$ min is easy

Optima over an M-convex set

Theorem (earlier and recent equivalences)

For an element m *of M-convex set* $\overset{....}{B}$*, the following are equivalent.*

- m *is dec-min*
- m *is inc-max*
- m *minimizes the square-sum* $\sum[x(s)^2 : s \in S]$
- m *minimizes the difference-sum* $\sum[|x(s) - x(t)| : s, t \in S]$
- m *minimizes the sum of the* k *largest components simultaneously for each* $k = 1, 2, \ldots, |S|$
- m *minimizes the total* a*-excess* $\sum[(x(s) - a)^+ : s \in S]$ *for each integer* a
- m *minimizes* $\sum \varphi(m(s))$ *for every strictly convex function* φ

Cheapest dec-min in-degree bounded orientations

$G = (V, E)$: undirected graph, with in-degree bounds (f, g)

given a cost $c(uv)$ and $c(vu)$ of both possible orientations of $uv \in E$, find a cheapest in-degree bounded orientation of G

reduces to : min-cost flows

find a cheapest dec-min in-degree bounded orientation

Theorem (2019+)

$\exists$ f^* and g^* with $f^*(v) \leq g^*(v) \leq f^*(v) + 1$ and $\exists$ a subset $E_0 \subseteq E$ with an orientation A_0 such that

an (f, g)-bounded orientation $D = (V, A)$ is dec-min (f, g)-bounded $\iff$ D is (f^*, g^*)-bounded and $A_0 \subseteq A$.

Describing dec-min extended semi-matchings

Recall:

$G = (S, T; E)$: bigraph, $f : (S \cup T) \to \mathbf{Z}_+$: lower bound, $g : (S \cup T) \to \mathbf{Z}_+$: upper bound, γ: positive integer

find an (f, g)-degree-bounded subgraph $F \subseteq E$ with γ edges such that the degree-vector $(d_F(s) : s \in S)$ on S (!!!) is decreasingly minimal

this is a special dec-min in-degree bounded orientation problem
$\Longrightarrow$ even the min-cost version is tractable

BUT . . .

if decreasing minimality of $d_F(v)$ is requested for the whole $S \cup T$ (or on any specified subset $Z \subseteq S \cup T$) ,
essentially new ideas are needed

Inc-max flow optimization on source-edges, I

$D = (V, A)$: digraph

$s \in V$: source node (with no entering arcs)

$t \in V$: sink node (with no leaving arcs)

$g : A \rightarrow \mathbf{R}_+$: non-negative rational-valued capacity function

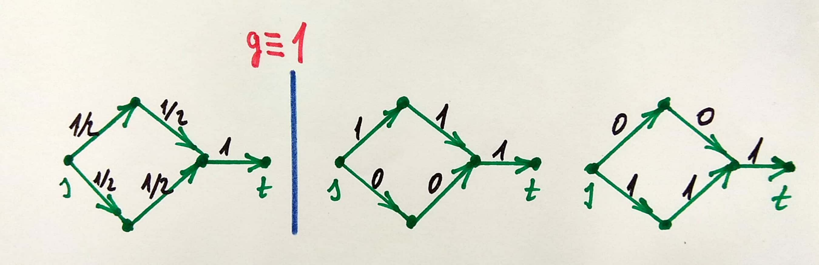
S_A: set of source-edges (= arcs leaving s)

$x : A \rightarrow \mathbf{R}_+$: a flow from s to t is **feasible** if $x \leq g$

flow amount of x: $\delta_x(s) = \widetilde{x}(S_A)$

max-flow: a feasible flow with maximum flow amount

Inc-max flow optimization on source-edges, II



the fractional inc-max flow on S_A two inc-max integral flows on S_A

Megiddo (1974, 1977) solved: find a (possibly fractional) max-flow x whose restriction to S_A is 'lexicographically optimal' (= increasingly maximal)

the (unique) optimal x may be fractional even if g is integer-valued

Discrete Megiddo-flows

(2018+) discrete version of Megiddo:

where g is integer-valued,
find an integral feasible max-flow z
whose restriction to S_A is increasingly maximal

known: given $D = (V, A)$ with source node s and sink node t,
the max-flows restricted to S_A span a base-polyhedron B

Strongly polynomial algorithm

the general strongly polynomial algorithm developed for finding a dec-min (= inc-max) element of an M-convex set $\dddot{B}$ can be applied:

in graph orientations, matroid optimizations, resource allocation, and discrete (Megiddo-type) inc-max flow problems

direct subroutines for supermodular function maximization are available via standard max-flow and matroid algorithms

Decreasingly minimal integer-valued flows

$D = (V, A)$: digraph

$m : V \to \mathbf{Z}$ with $\widetilde{m}(V) = 0$

$z : A \to \mathbf{Z}$: m-flow if $\varrho_z(v) - \delta_z(v) = m(v)$ for every $v \in V$

$f : A \to \mathbf{Z} \cup \{-\infty\}$: lower bound
$g : A \to \mathbf{Z} \cup \{+\infty\}$: upper bound ($f \leq g$)

(f, g)-bounded m-flow z: $f \leq z \leq g$

$F \subseteq A$: specified subset of edges

z F-dec-min: the largest z-value on F is as small as possible, within this, the second largest z-value on F is as small as possible, etc.

$Q :=$ set of F-dec-min (f, g)-bounded m-flows

Decreasingly minimal flows: a special case

Kaibel + Onn + Sarrabezolles (2015) solved:
find an uncapacitated integral dec-min *st*-flow of given flow-amount M

original version: find M *st*-paths so that the largest burden of an edge is minimal, within this, the second largest burden of an edge is minimal, etc.

burden of e: the number of dipaths using e

(lucky case is when $\exists$ M edge-disjoint *st*-paths)

Kaibel + Onn + Sarrabezolles:
polynomial algorithm for fixed M
(but not polynomial when M is not fixed)

The set of F-dec-min m-flows

the set of (f, g)-bounded integral m-flows is not M-convex, in general

hence dec-min is not the same as inc-max

Theorem (2018-19+)

$\exists$ *integer-valued functions* f^* *and* g^* *on* A *with* $f \leq f^* \leq g^* \leq g$ *such that* $z \in \ddot{Q}$ *is* F*-dec-min* $\Longleftrightarrow$ z *is an integral* (f^*, g^*)*-bounded* *m-flow. Moreover, the box* $T(f^*, g^*)$ *is narrow on* F*:* $0 \leq g^*(e) - f^*(e) \leq 1$ *for every* $e \in F$.

2019+: strongly polynomial algorithm to compute (f^*, g^*)

2019+: strongly polynomial algorithm to compute a min-cost integral feasible m-flow which is dec-min on F

Extensions

for mixed graphs, dec-min strong orientation $\neq$ inc-max strong orientation

reason: the set of in-degree vectors of strong orientations of a mixed graph is not an M-convex set, in general,
but the intersection of two M-convex sets

Edmonds: the intersection $B := B_1 \cap B_2$ of two integral base-polyhedra is an integral polyhedron

different problems:
find a dec-min element of $\ddddot{B}$
find a square-sum minimizer element of $\ddddot{B}$

A new min-max theorem on square-sum

Theorem (2018+)

Let $B_1 = B'(p_1)$ *and* $B_2 = B'(p_2)$ *be integral base-polyhedra defined by supermodular functions* p_1 *and* p_2 *for which* $B = B_1 \cap B_2$ *is non-empty. Then*

$$\min \{\sum [m(s)^2 : s \in S] : m \in \ddot{\ddot{B}}\} =$$

$$\max \ \{\hat{p}_1(\pi_1) + \hat{p}_2(\pi_2) - \sum_{s \in S} \lfloor \tfrac{\pi_1(s)+\pi_2(s)}{2} \rfloor \lceil \tfrac{\pi_1(s)+\pi_2(s)}{2} \rceil : \pi_1, \pi_2 \in \mathbf{Z}^S\}.$$

the proof uses tools from Discrete convex analysis difficulties:

- dec-min $\neq$ inc-max
- local improvement does not suffice

more general framework: submodular flows

Theorem (2018+)

Given a feasible submodular flow polyhedron Q ,

$\exists$ *a small box* T *and a face* F *of* Q *such that* $z \in \ddot{\ddot{Q}}$ *is dec-min*

$\Longleftrightarrow$

$z \in F \cap T$.

$\exists$ polynomial algorithm