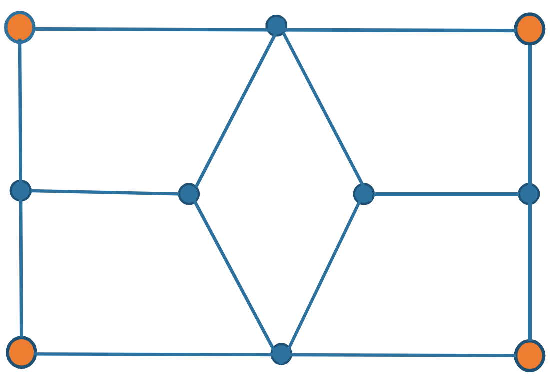
Mader's Theorem on Edge-Disjoint T-Paths

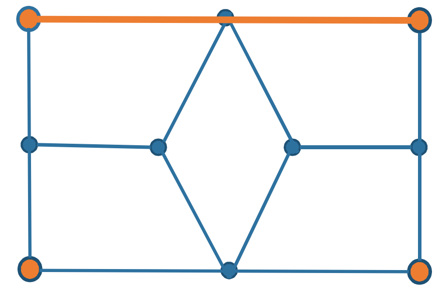
Satoru Iwata (University of Tokyo)

Joint work with Yu Yokoi (NII)

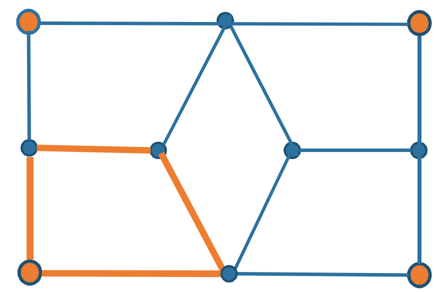
T-Paths

Graph $G = (V, E)$, $T \subseteq V$





A T-path

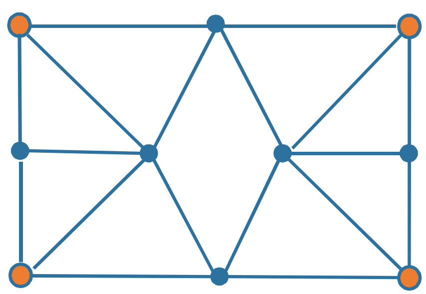


Not a T-path

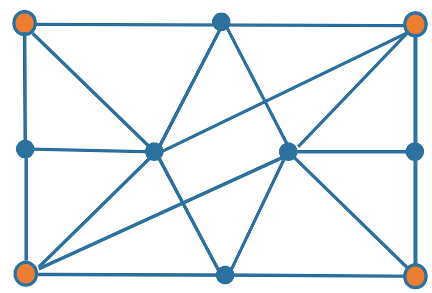
$\mu(G, T)$: # edge-disjoint T-paths

T-Paths

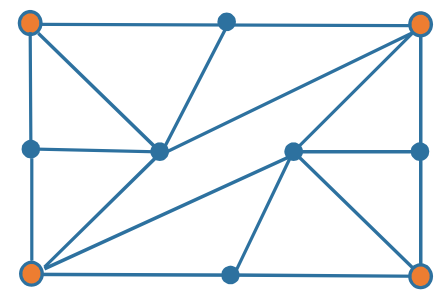
Graph $G = (V, E)$, $T \subseteq V$



$\mu(G, T) = 6$



$\mu(G, T) = 7$



$\mu(G, T) = 6$

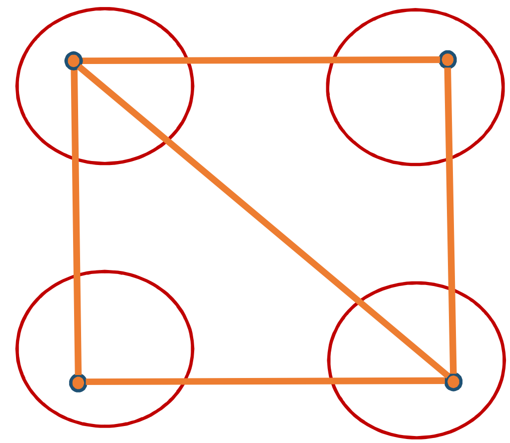
$\mu(G, T)$: # edge-disjoint T-paths

An Upper Bound

$\mathcal{X} = \{X_s\}_{s \in T}$: T-subpartition

$$X_s \cap T = \{s\}, \forall s \in T$$
$$X_s \cap X_t = \emptyset, \forall s, t \in T$$

$$\mu(G, T) \leq \frac{1}{2} \sum_{s \in T} d(X_s)$$



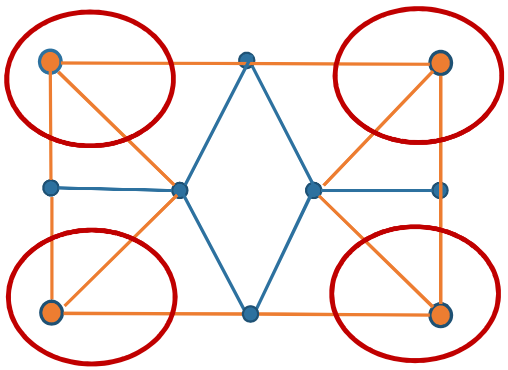
Theorem [Lovász (1976), Cherkassky (1977)]

(G, T): Inner Eulerian ($d(v)$: even, $\forall v \in V \setminus T$)

$$\Longrightarrow \qquad \mu(G, T) = \frac{1}{2} \min_{\mathcal{X}} \sum_{s \in T} d(X_s)$$

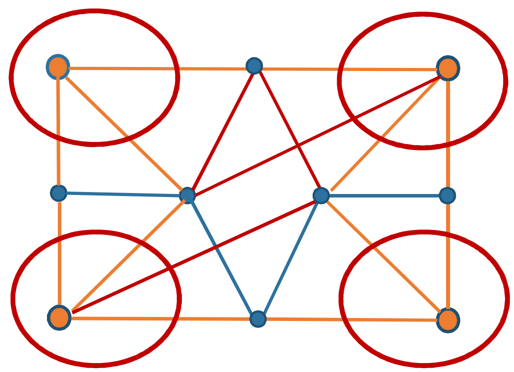
An Upper Bound

$$\sum_{s \in T} d(X_s) = 12$$



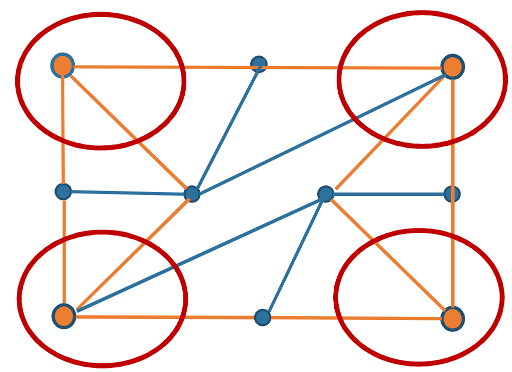
$$\mu(G,T) = 6$$

$$\sum_{s \in T} d(X_s) = 14$$



$$\mu(G,T) = 7$$

$$\sum_{s \in T} d(X_s) = 14$$



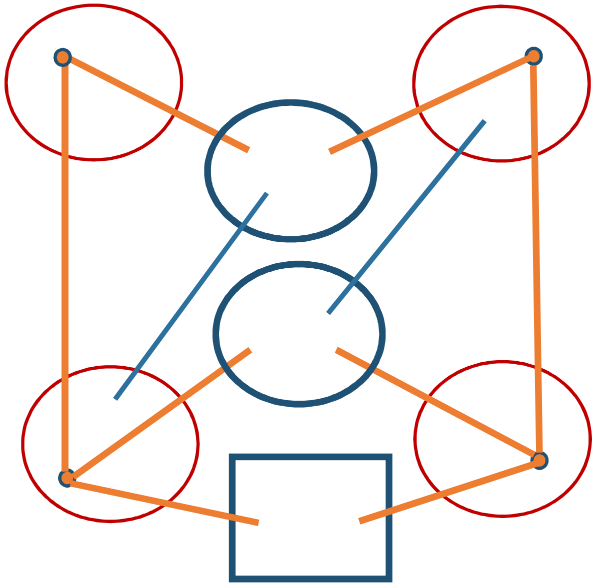
$$\mu(G,T) = 6$$

A Tighter Upper Bound

$\mathcal{X} = \{X_s\}_{s \in T}$: T-subpartition

$\mathrm{odd}(G \setminus \mathcal{X})$: # odd degree components in $G \setminus \mathcal{X}$

$$\mu(G,T) \le \kappa(\mathcal{X}) := \frac{1}{2}\left[\sum_{s \in T} d(X_s) - \mathrm{odd}(G \setminus \mathcal{X})\right]$$



Mader's Theorem

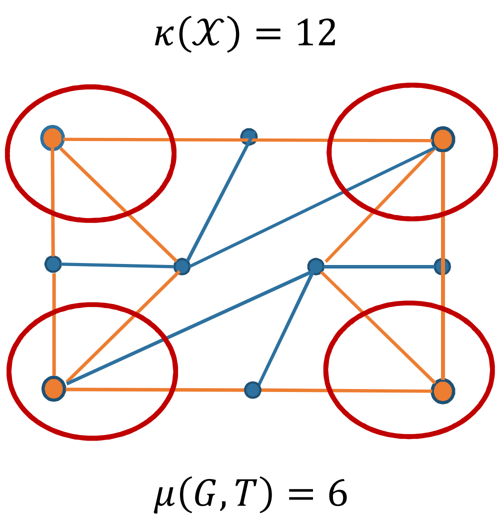
$\mathcal{X} = \{X_s\}_{s \in T}$: T-subpartition

$\mathrm{odd}(G \setminus \mathcal{X})$: # odd degree components in $G \setminus \mathcal{X}$

$$\mu(G,T) \le \kappa(\mathcal{X}) := \frac{1}{2}\left[\sum_{s \in T} d(X_s) - \mathrm{odd}(G \setminus \mathcal{X})\right]$$

Theorem [Mader 1978]

$$\mu(G,T) = \frac{1}{2}\min_{\mathcal{X}}\left[\sum_{s \in T} d(X_s) - \mathrm{odd}(G \setminus \mathcal{X})\right]$$



Original Papers by Mader

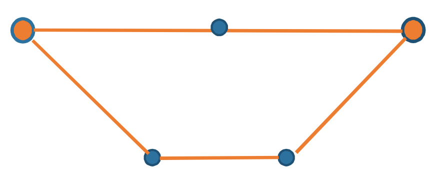
- W. Mader: Über die Maximalzahl kantendisjunkter A-Wege, *Archiv. Math.*, 30 (1978), pp.325--336.

edge-disjoint

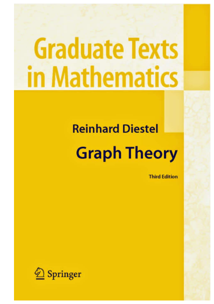
- W. Mader: Über die Maximalzahl kreuzungsfreier H-Wege, *Archiv. Math.*, 31 (1978), pp.387--402.

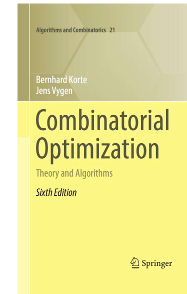
cross-free

openly disjoint

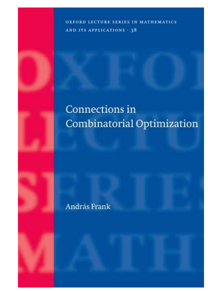


Proofs from Books

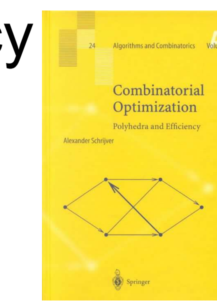




- R. Diestel: Graph Theory.
  --- Section for Mader's theorems.
  --- No Proofs.
- B. Korte & J. Vygen: Combinatorial Optimization: Theory and Algorithms
  --- No Mentions.

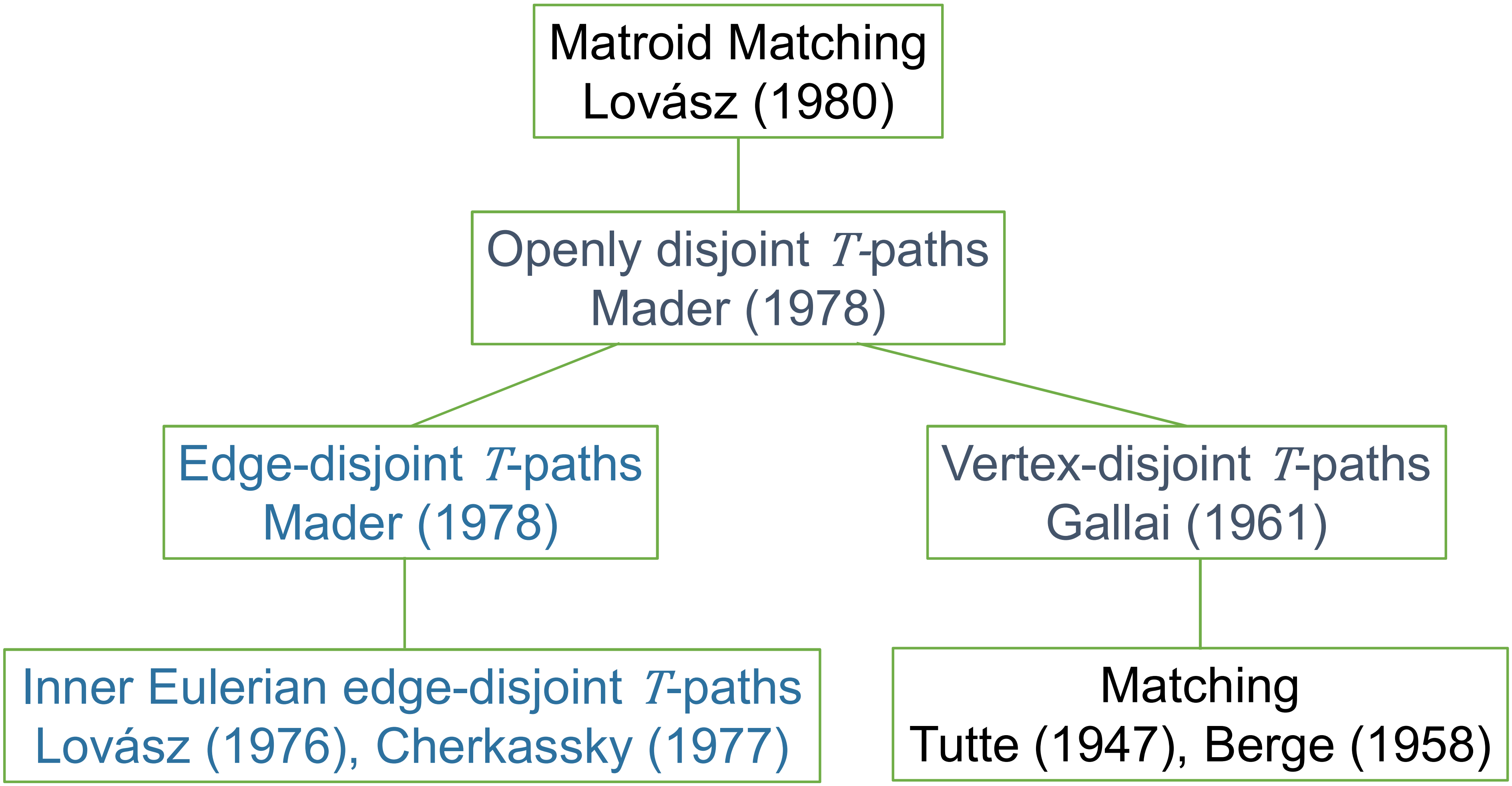


- A. Frank: Connections in Combinatorial Optimization.
  --- Theorem of Lovász and Cherkassky.



- A. Schrijver: Combinatorial Optimization: Polyhedra and Efficiency
  --- A short proof on openly disjoint T-paths.
  --- Reduction via line graphs.

Hierarchy of Frameworks



Previous Works

- Lovász (1980): Reduction to matroid matching.
- Karzanov (1993, 1997): Minimum cost edge-disjoint T-paths.
- Schrijver (2001): Short proof for openly disjoint T-paths.
- Schrijver (2003): Reduction to linear matroid parity.
- Keijsper, Pendavingh, Stougie (2006):
  LP formulation of maximum edge-disjoint T-paths.
- Hirai and Pap (2014):
  Weighted maximization with tree metric.

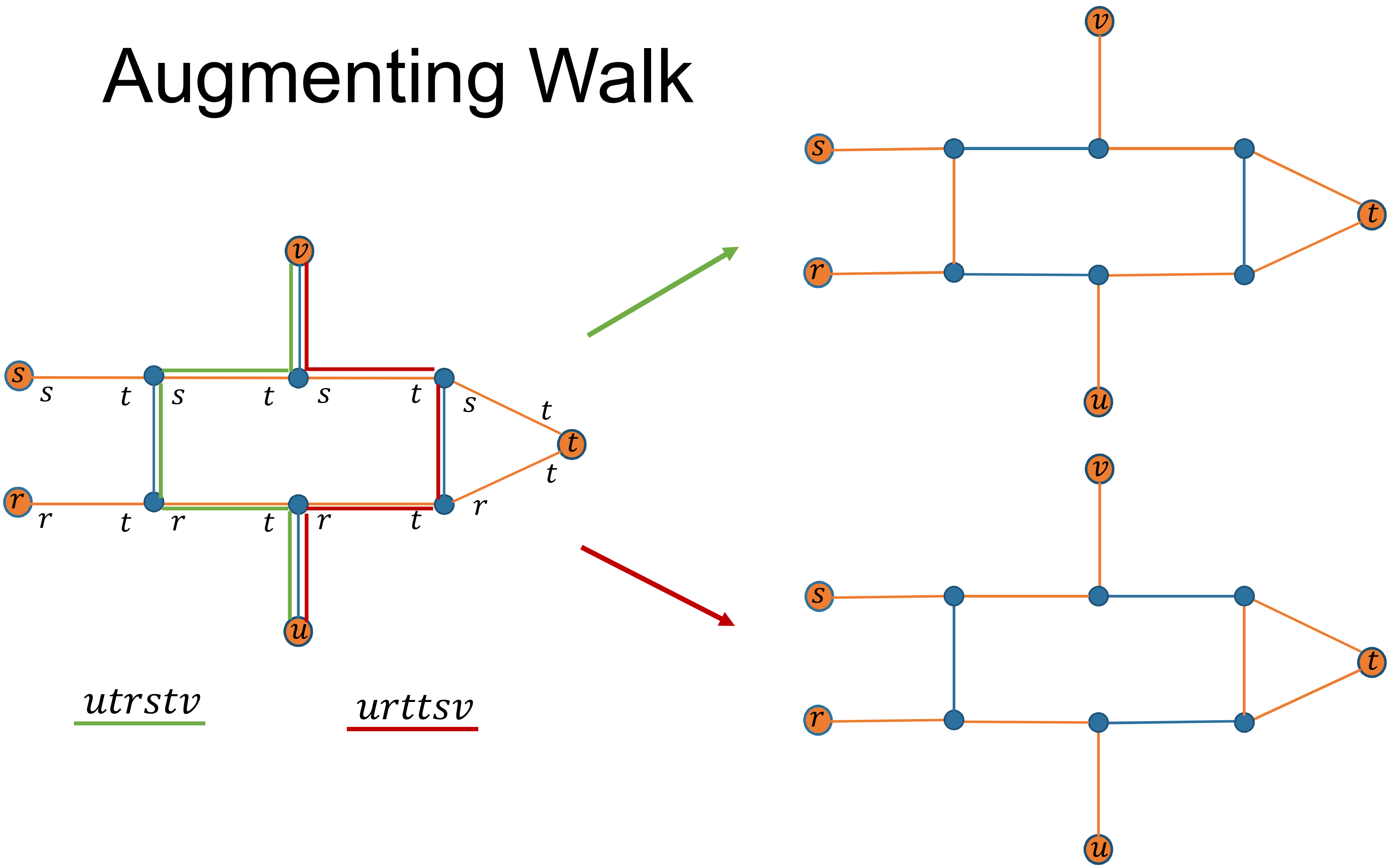
Our Contribution

- A constructive proof of Mader's theorem on edge-disjoint T-paths.

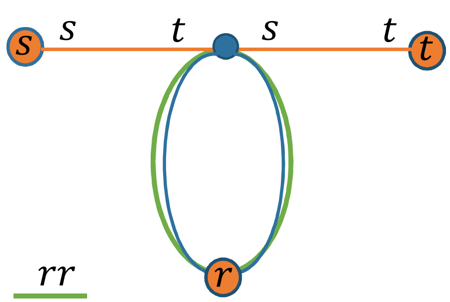
- A combinatorial algorithm for finding maximum edge-disjoint T-paths.

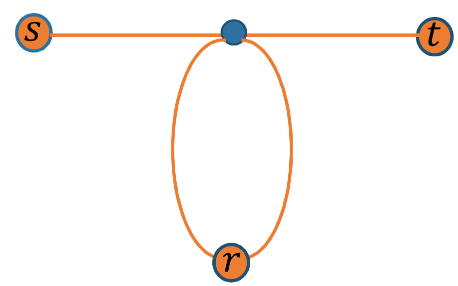
  Running time: $O(|V| \cdot |E|^2)$.

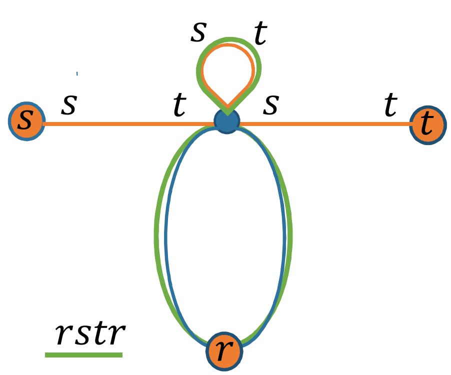
Augmenting Walk



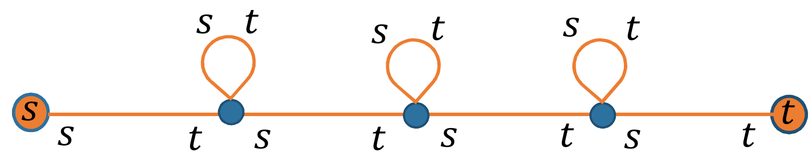
Augmenting Walk



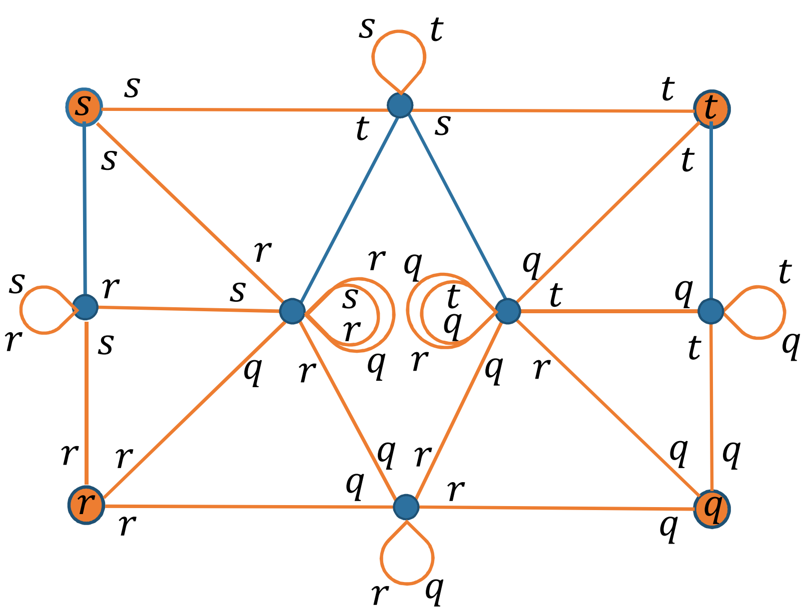




Augmenting Walk



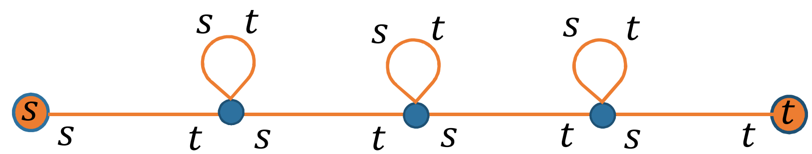
Auxiliary Labeled Graph



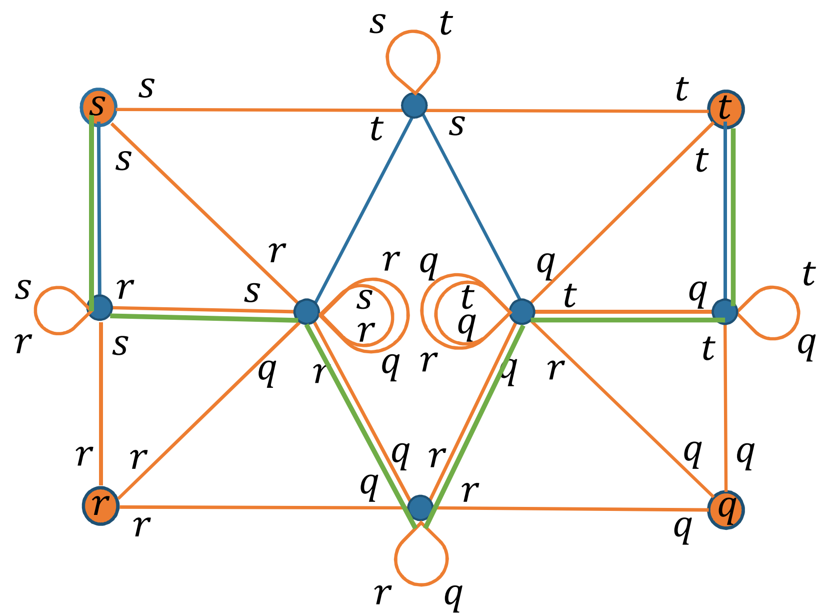
Augmenting walk:

- Between terminals
- No consecutive symbols
- Uses edge at most once,
- Uses selfloop at most once,
- Uses edge at most twice, at most once in each direction

Augmenting Walk



Auxiliary Labeled Graph



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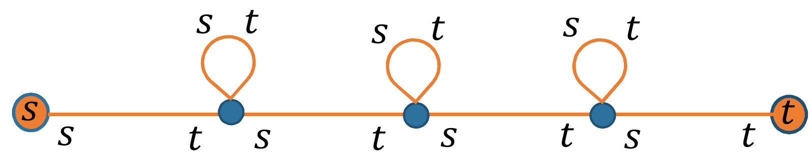
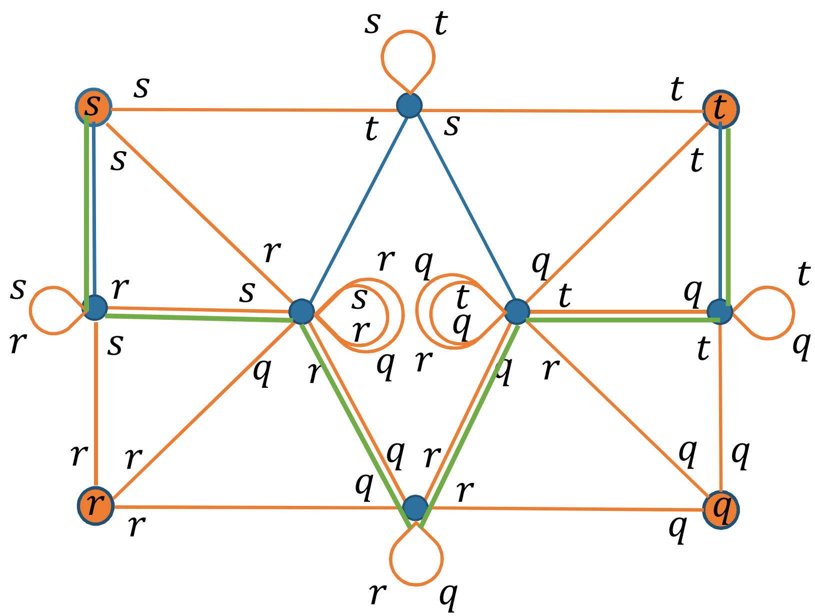
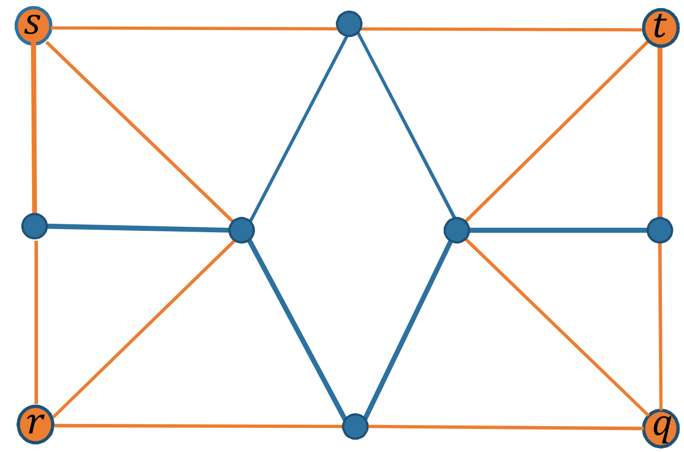
Augmenting Walk



Auxiliary Labeled Graph

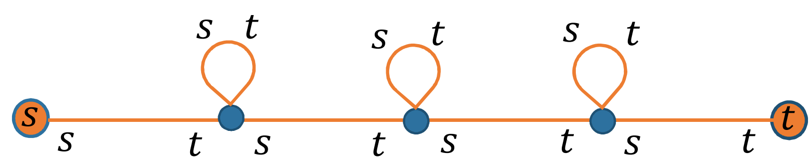


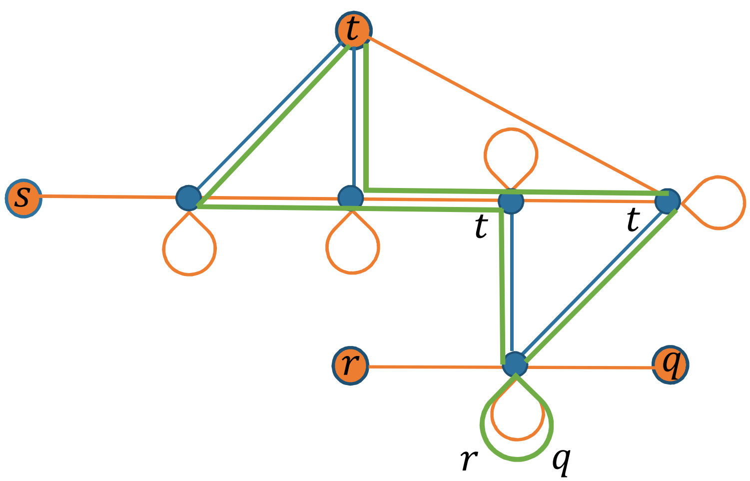


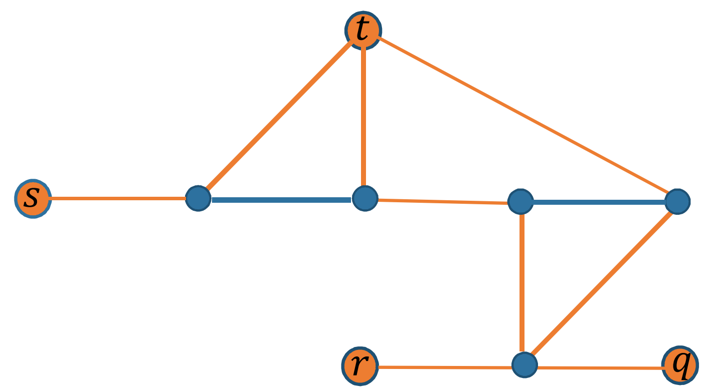
Symmetric Difference

Augmenting Walk

Auxiliary Labeled Graph







Symmetric Difference

Augmentation

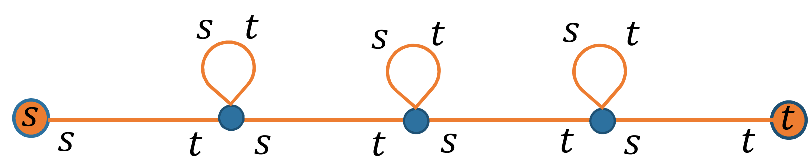
$\mathcal{P} := (P_1, \dots, P_k)$ Edge-disjoint T-paths

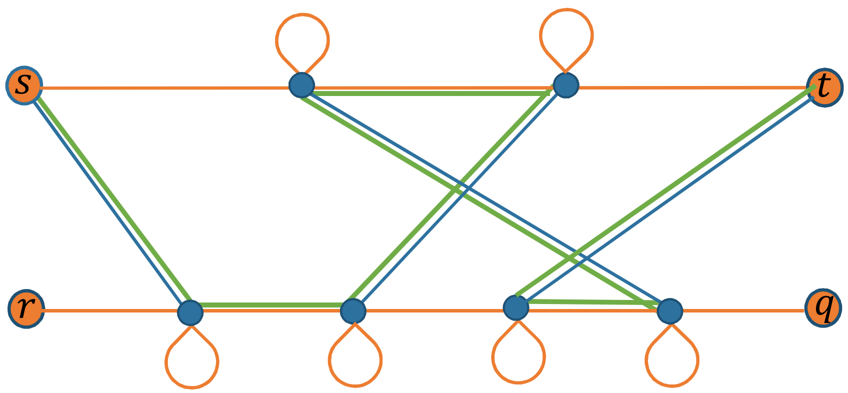
∃ Augmenting Walk in the Auxiliary Labeled Graph

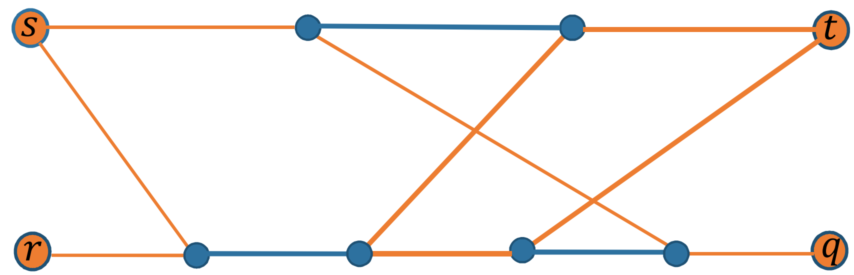
$\Longrightarrow \exists\ k+1$ Edge-disjoint T-paths

Augmentation

Auxiliary Labeled Graph



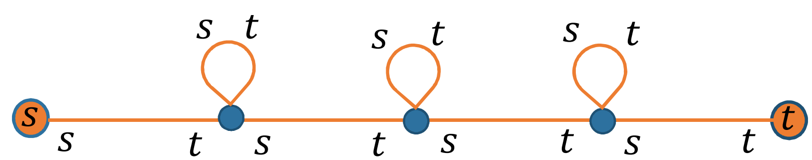


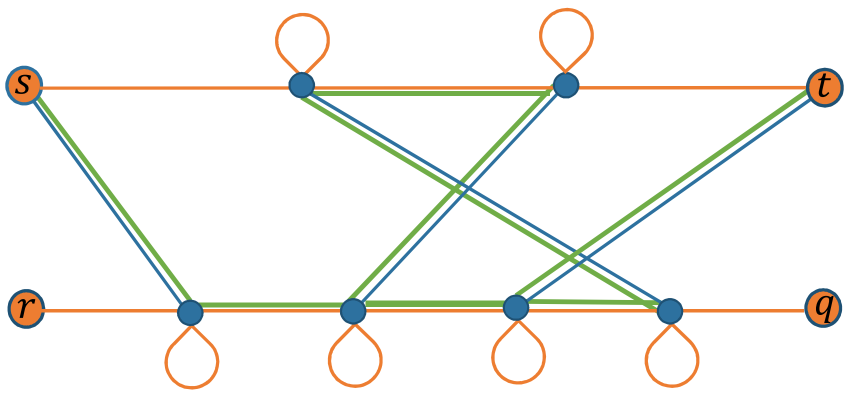


Symmetric Difference

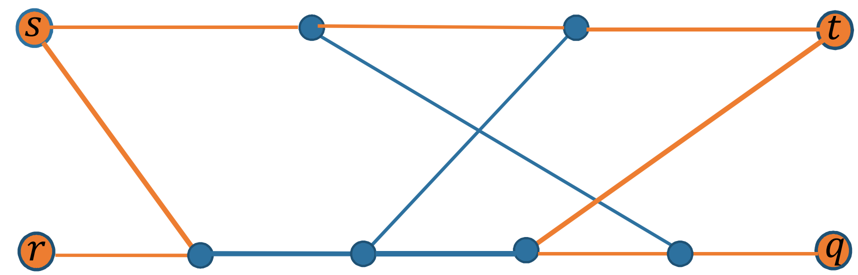
Augmentation

Auxiliary Labeled Graph



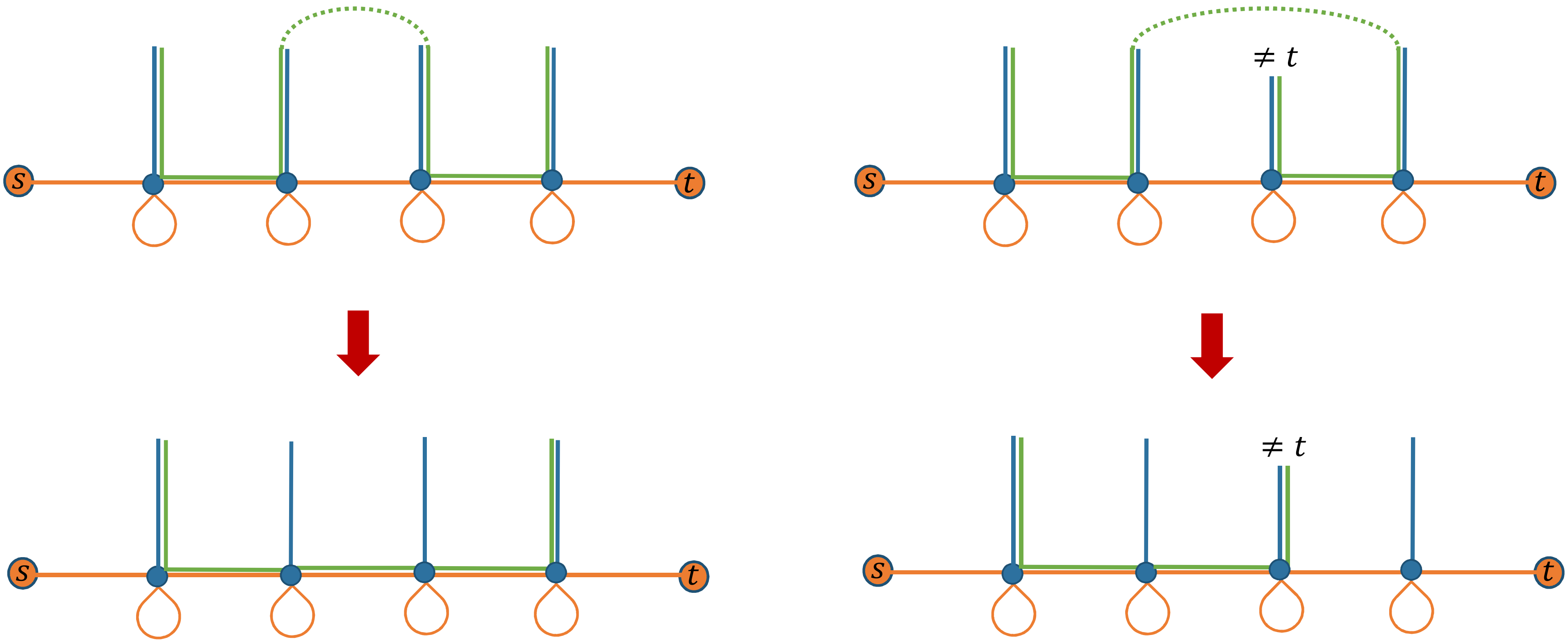


Shortcut

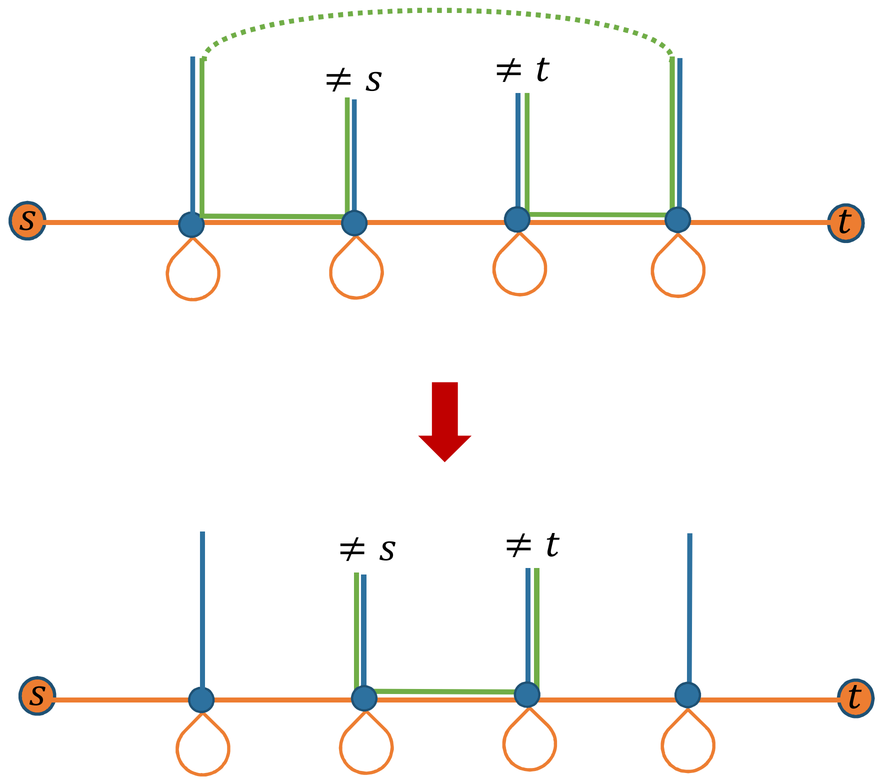


Symmetric Difference

Shortcut Operations



Shortcut Operations



Validity of Augmentation

$\mathcal{P} \coloneqq (P_1, \ldots, P_k)$ Edge-disjoint T-paths

Q: Augmenting walk w/o shortcuts

$\Longrightarrow H := (V, E(\mathcal{P}) \Delta E(Q))$ has $k+1$ edge-disjoint T-paths.

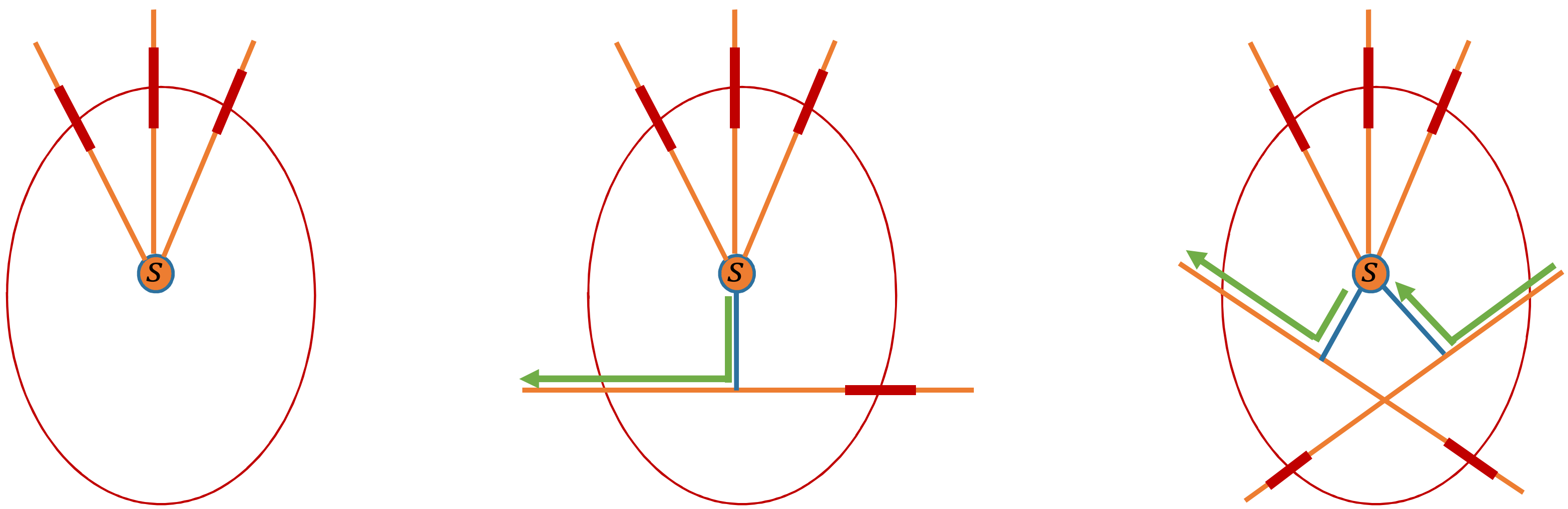
H: Inner Eulerian

→ Apply the theorem of Lovász & Cherkassky

$$\mu(H, T) = \frac{1}{2} \min_{\mathcal{X}} \sum_{s \in T} d(X_s)$$

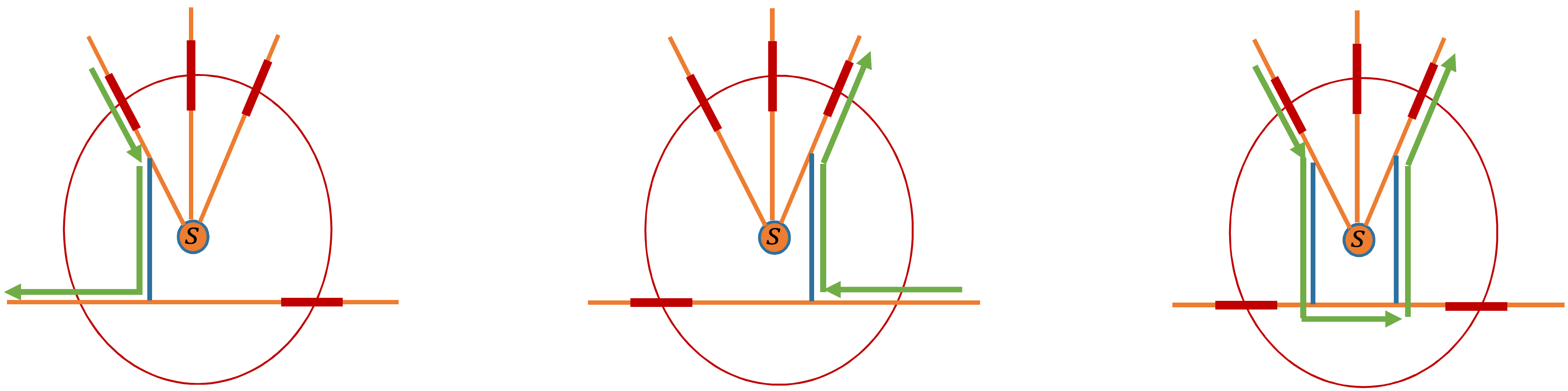
Validity of Augmentation

$$\forall s \in T, \forall X_s \subseteq V, X_s \cap T = \{s\} \Longrightarrow d_H(X_s) \geq d_H(s)$$

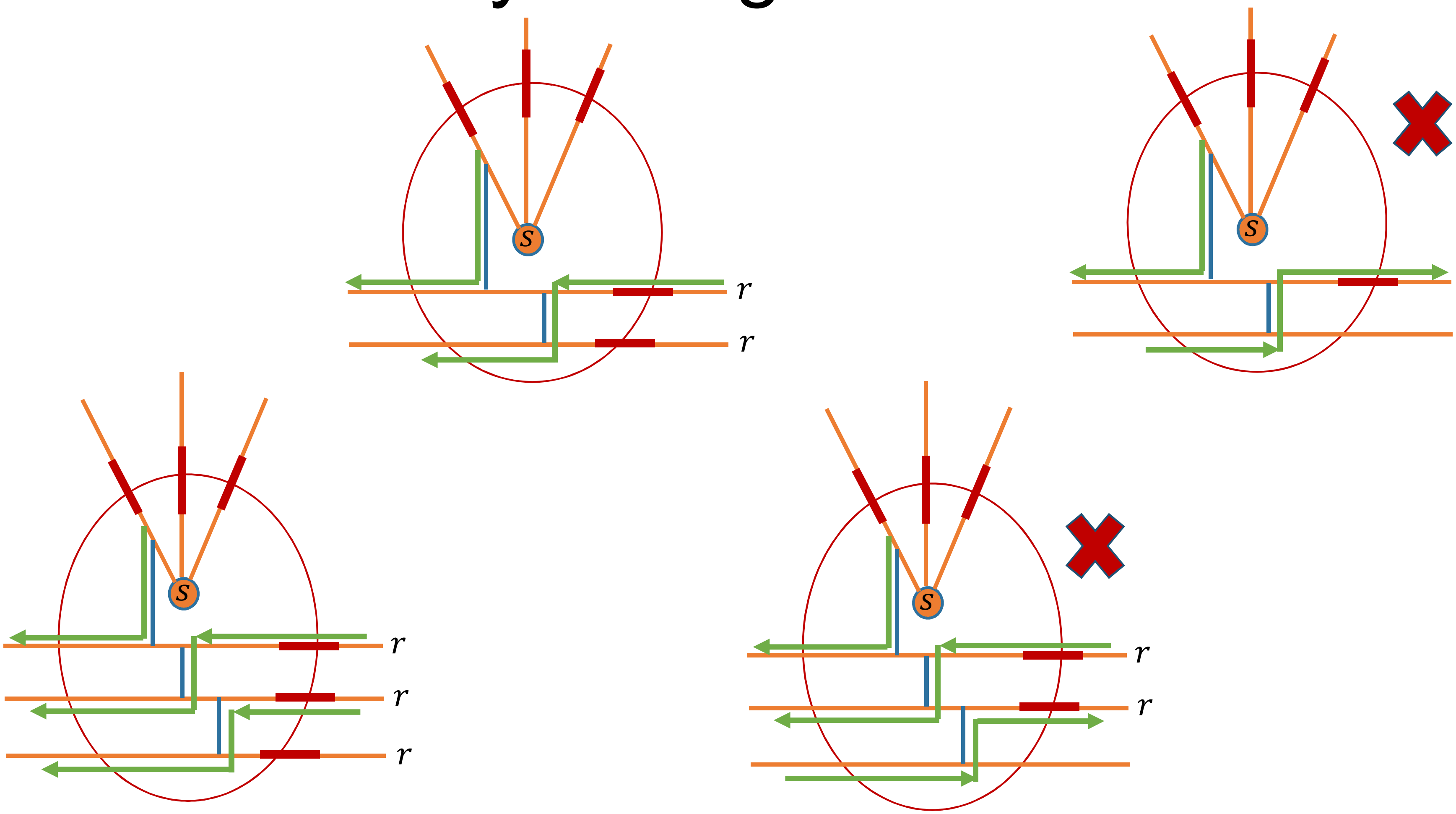


$d_Q(s) = 0$ $d_Q(s) = 1$ $d_Q(s) = 2$

Validity of Augmentation



Validity of Augmentation



Tightness

$\mathcal{P} \coloneqq (P_1, \dots, P_k)$ Edge-disjoint T-paths

No Augmenting Walks in the Auxiliary Labeled Graph

$\Longrightarrow \exists \mathcal{X}$: T-subpartition such that $\kappa(\mathcal{X}) = k$.

$$\kappa(\mathcal{X}) \coloneqq \frac{1}{2}\left[\sum_{s \in T} d(X_s) - \mathrm{odd}(G \setminus \mathcal{X})\right]$$

Tightness

$\mathcal{P} := (P_1, \dots, P_k)$ Edge-disjoint T-paths

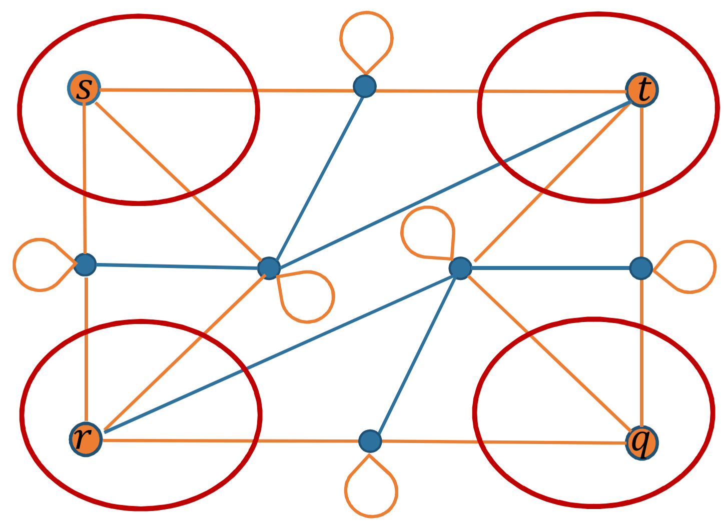
$\lambda_Q(v)$: The last symbol in the admissible walk Q from T to v.

$\Lambda(v) := \{ \lambda_Q(v) \mid Q\text{: admissible walk from } T \text{ to } v \}$.

$X_s := \{ v \in V \mid \Lambda(v) = \{s\} \} \quad (s \in T)$

$\mathcal{X} := \{ X_s \mid s \in T \}$

$\kappa(\mathcal{X}) = k$

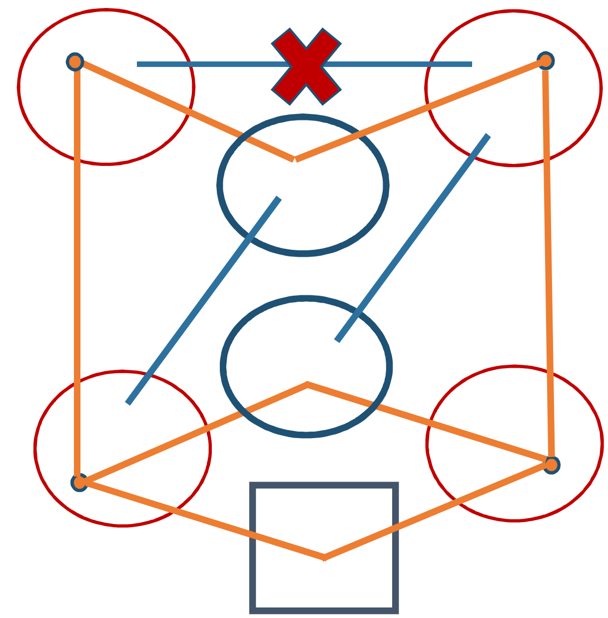


Tightness

No —— edge between X_s and X_t for $s \neq t$.

A T-path P_j between s and t is disjoint from X_r with $r \notin \{s, t\}$.

At most one —— edge leaves a connected component of $G \setminus \mathcal{X}$.



$$\sum_{s \in T} d(X_s) = 2k + \mathrm{odd}(G \setminus X)$$

$$k := \frac{1}{2}\left[\sum_{s \in T} d(X_s) - \mathrm{odd}(G \setminus \mathcal{X})\right]$$

Summary

- A constructive proof of Mader's theorem on edge-disjoint T-paths.

- A combinatorial algorithm for finding maximum edge-disjoint T-paths.

S. Iwata and Y. Yokoi: A blossom algorithm for maximum edge-disjoint T-paths, METR 2019-16.
https://www.keisu.t.u-tokyo.ac.jp/research/techrep/y2019/

Future Directions

- A combinatorial algorithm for minimum cost edge-disjoint T-paths.
- A combinatorial algorithm for the integer free multiflow problem.
- A combinatorial algorithm for maximum openly disjoint T-paths w/o reduction to matroid parity.