Minimum cuts via Breadth-First search

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Outline

- Minimum s-t cut in digraphs (folklore)
- Multiway-cuts in undirected graphs (folklore)
- Multiway-cuts in digraphs (Chekuri & Madan)
- Multicuts in undirected graphs (Calinescu, Karloff & Rabani)

Minimum s-t cut problem

Given digraph G=(V,A), with nonnegative costs/capacities on arcs, a source s and a sink t, find minimum cost arc set blocking all s-t paths

$$z_{SP} \;=\; \min \sum_{arcs\; uv} w_{uv}\, x_{uv}$$

s.t.

$$\textstyle\sum_{uv \in P} x_{uv} \geq 1 \;\; \forall\, P \;\text{ s-t paths}$$

$$x_{uv} \geq 0 \;\; \forall uv \;\text{ arcs}$$

Compact LP formulation

$$z_{SP} = \min \sum_{arcs\ uv} w_{uv}\, x_{uv}$$

s.t.

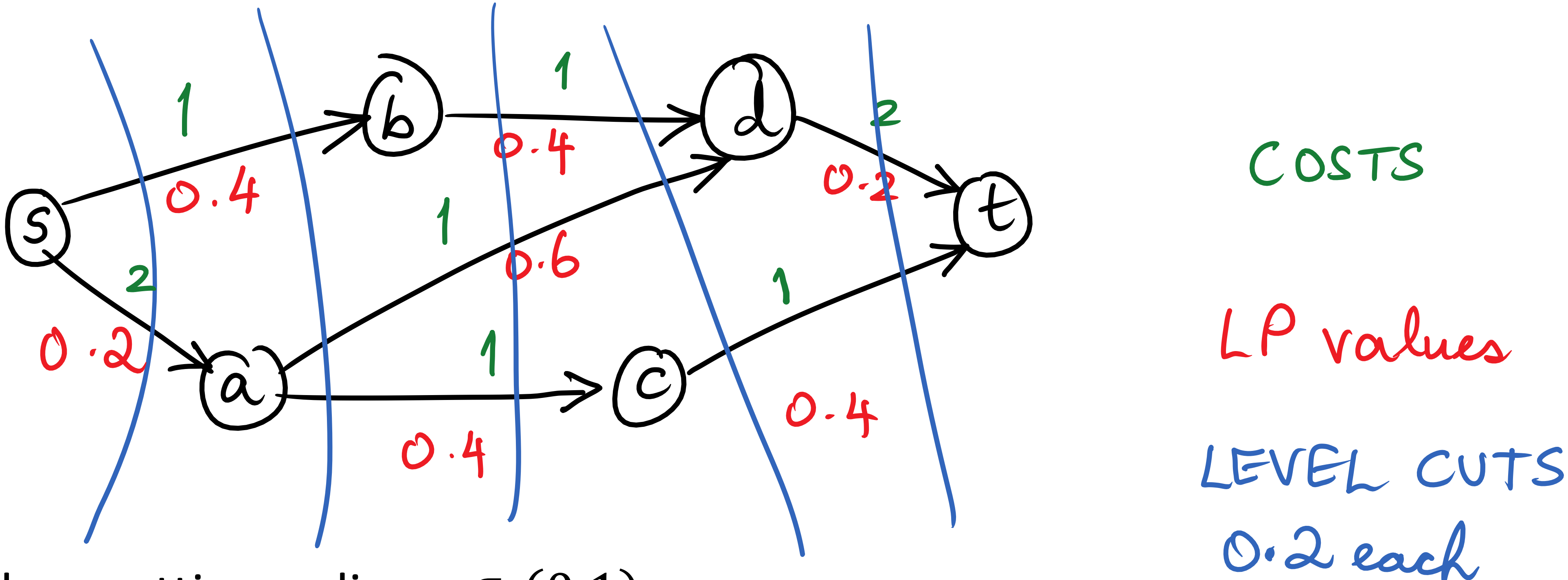
$$d_v \leq d_u + x_{uv} \quad \forall uv$$

$$d_s = 0$$

$$d_t \geq 1$$

$$x_{uv} \geq 0 \quad \forall uv$$

Djikstra's algorithm to generate level cuts



- Pick random cutting radius $r \in (0,1)$
- Remove edges leaving the ball $B(s,r)$ of nodes contained within a radius of r around the source s with respect to the distances x_{uv} on arcs $uv \in A(G)$

Every level cut is an optimal cut!

- Pick random cutting radius $r \in (0,1)$
- Cut edges leaving the ball $B(s,r)$ of nodes contained within a radius of r around the source s with respect to the distances x_{uv} on arcs $uv \in A(G)$
- Output $\delta^+\big(B(s,r)\big) \stackrel{\text{def}}{=} level\ cut_r$

  Claim: $P(uv \in cut) \leq x_{uv}$

  Corollary: $E\big(w(cut)\big) \leq \sum_{uv} w_{uv} P(uv \in cut) \leq z_{SP}$

$$w(mincut) \leq \min_{levels} w(level\ cut) \leq E\big(w(cut)\big) \leq z_{SP} \leq w(mincut)$$

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Multiway cut in undirected graphs

Given undirected graph G=(V,E), with nonnegative costs on edges, a source set S = $\{s_1, \dots, s_k\}$, find minimum cost edge set blocking all $s_i - s_j$ paths

$$z_{UMWC} = \min \sum_{edges\ uv} w_{uv}\, x_{uv}$$

s.t.

$$\sum_{uv \in P} x_{uv} \geq 1 \;\; \forall\, P\ s_i - s_j \text{ paths}$$

$$x_{uv} \geq 0 \;\; \forall uv \text{ edges}$$

Natural generalization of level cuts

- Pick random cutting radius $r \in \left(0, \frac{1}{2}\right)$
- Cut edges leaving the ball $B(s_i, r)$ of nodes contained within a radius of r around the each source s_i with respect to the distances x_{uv} on edges $uv \in E(G)$
- Output all but the heaviest cut in $\cup_i \ \delta\big(B(s_i, r)\big) \stackrel{\text{def}}{=} level\ cut_r$

A 2-approximation algorithm

Claim: $P(uv \in cut) \leq \frac{x_{uv}}{\frac{1}{2}} = 2x_{uv}$

Corollary: $E\big(w(cut)\big) \leq 2z_{UMWC}$

$$\min_{levels\ r} w(level\ cut_r) \leq \left(1 - \frac{1}{k}\right) E\big(w(cut)\big)$$
$$\leq 2\left(1 - \frac{1}{k}\right) z_{UMWC}$$
$$\leq 2\left(1 - \frac{1}{k}\right) w(\min multiway\ cut)$$

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Multiway cuts in digraphs

Given directed graph G=(V,A), with nonnegative costs on arcs, a source set S = $\{s_1, \ldots, s_k\}$, find minimum cost arc set blocking all $s_i \rightarrow s_j$ paths for all ordered pairs of sources $i \neq j$.

Note: Min multiway cut for S = $\{s_1, s_2\}$ is NP-hard so does not specialize to regular min-cut (Also need to cut all reverse paths)

Multiway cuts in digraphs

Given directed graph G=(V,A), with nonnegative costs on arcs, a source set S = $\{s_1, \ldots, s_k\}$, find minimum cost arc set blocking all $s_i \to s_j$ paths for all ordered pairs of sources $i \neq j$.

- In digraphs, node weights can be represented as arc weights by dividing nodes
- Generalizes node-weighted multiway cut in undirected graphs:

Given **undirected** graph G=(V,E), a source set S = $\{s_1, \ldots, s_k\}$, and nonnegative costs on non-source ***nodes***, find minimum cost node set blocking all $s_i - s_j$ paths

- In undirected graphs, node weights generalize edge weights by subdividing

Multiway cuts in digraphs

Given directed graph G=(V,A), with nonnegative costs on arcs, a source set S = $\{s_1, \ldots, s_k\}$, find minimum cost arc set blocking all $s_i \rightarrow s_j$ paths for all ordered pairs of sources $i \neq j$.

Theorem (Naor-Zosin, FOCS'97): 2-approximation for multiway cuts in digraphs by exactly rounding a *relaxed multiway flow* relaxation which is within factor 2 of natural relaxation

Multiway cuts in digraphs (Chekuri & Madan, SODA '16)

$$z_{UMWC} = \min \sum_{edges\, uv} w_{uv}\, x_{uv}$$

s.t.

$$\textstyle\sum_{uv \in P} x_{uv} \geq 1 \;\; \forall\, P\; s_i \to s_j \text{ paths}$$

$$x_{uv} \geq 0 \;\; \forall uv \text{ arcs}$$

Theorem: Level-cutting algorithm on the above LP gives a 2-approximation

Level cuts – Attempt 1

- Pick random cutting radius $r \in (0,1\ )$
- Cut arcs leaving the ball $B(s_i, r)$ of nodes contained within a radius of r around the each source s_i with respect to the distances x_{uv} on arcs $uv \in A(G)$
- Output $\cup_i\ \delta^+\big(B(s_i, r)\big) \stackrel{\text{def}}{=} level\ cut_r$ for a random r
- Argue no arc is overused by more than factor 2 in expectation?

Level cuts – Attempt 1 fails

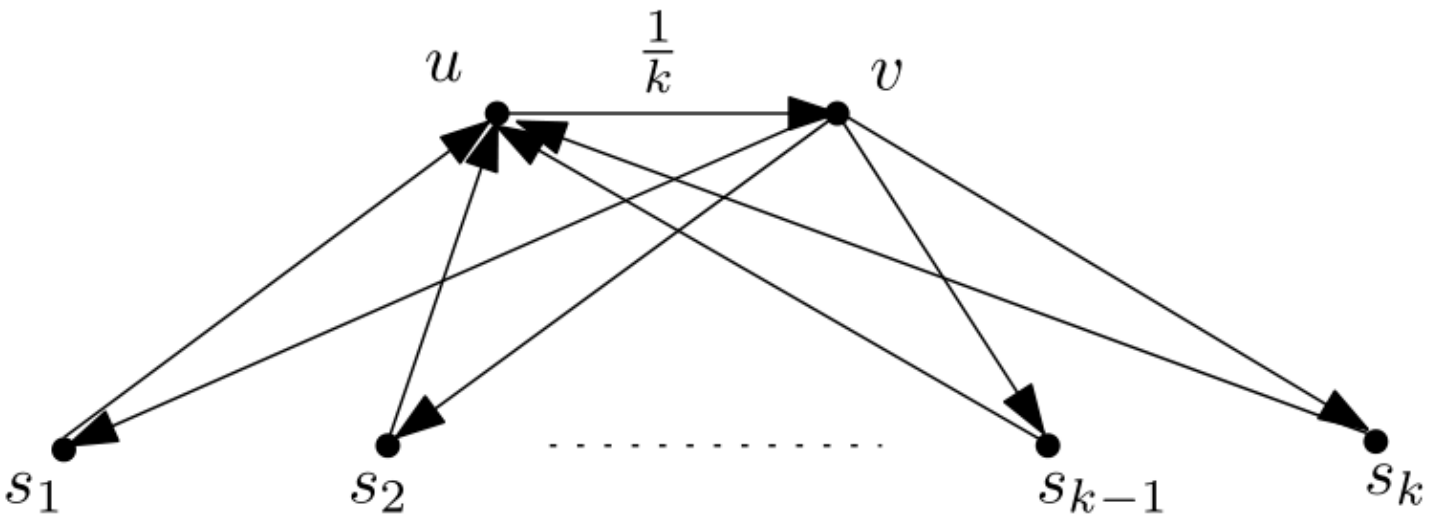
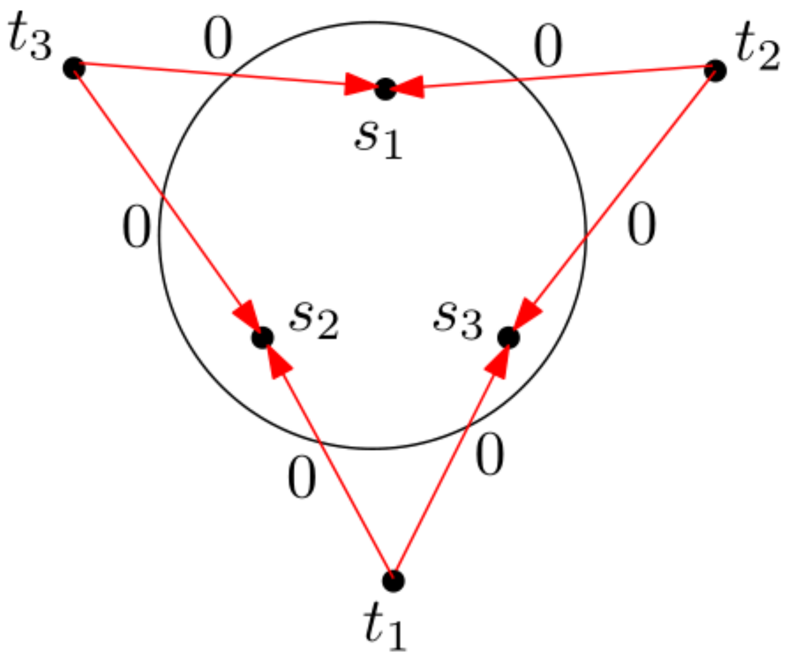


Figure 8: Edge $s_i u$ has length $(i-1)/k$, edge $v s_i$ has length 1 and edge uv has length $1/k$. Cost of edge uv is 1. Rest of edges have cost 0.

Level cuts of Chekuri and Madan



Algorithm 1 Rounding for DIR-MC

Given a feasible solution $\mathbf{x}$ to DIR-MC-REL

Add new vertices $t_1, \ldots, t_k$, edges (t_i, s_j) for all $i \neq j$ and set $x(t_i, s_j) = 0$

Pick $\theta \in (0, 1)$ uniformly at random

$C = \cup_{i=1}^{k} \delta^+(B(t_i, \theta))$

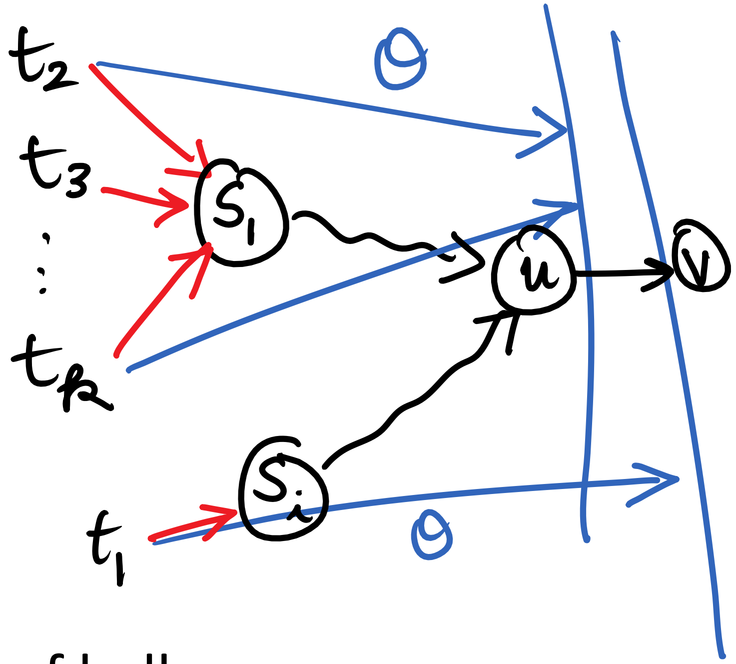
Return C

A 2-approximation algorithm

For arc uv order sources so that

$$d(s_1,u) \leq d(s_2,u) \leq \cdots \leq d(s_k,u)$$

Note that $d(t_2,u) = d(t_3,u) = \cdots = d(t_k,u) = d(s_1,u)$
If one of these balls cut uv then all of them do



Thus uv is either cut by the ball around t_1 or by the above set of balls.

$$P(uv \in cut) \leq 2x_{uv}$$

$$\min_{levels\ r} w(level\ cut_r) \leq 2w(\min multiway\ cut)$$

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Multicuts in undirected graphs

Given undirected graph G=(V,E), with nonnegative costs on edges, and source-sink pairs = $\{(s_1, t_1), \ldots, (s_k, t_k)\}$, find minimum cost edge set blocking all $s_i - t_i$ paths

$$z_{UMC} = \min \sum_{edges\, uv} w_{uv}\, x_{uv}$$

s.t.

$$\sum_{uv \in P} x_{uv} \geq 1 \;\; \forall\, P\; s_i - t_i \text{ paths}$$

$$x_{uv} \geq 0 \;\; \forall uv \text{ edges}$$

Level cut algorithm – attempt 1

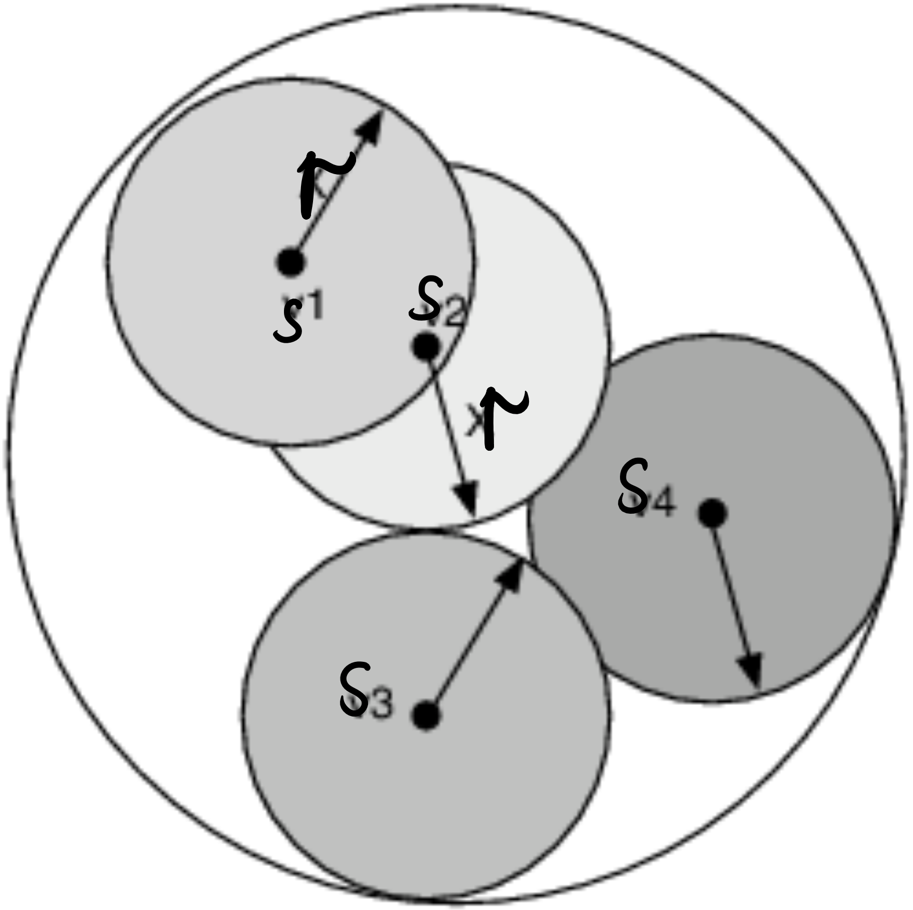
- Pick random cutting radius $r \in (0,1)$
- Cut edges leaving the ball $B(s_i, r)$ of nodes contained within a radius of r around the each source s_i with respect to the distances x_{uv} on edges $uv \in E(G)$
- Output $\cup_i \ \delta\big(B(s_i, r)\big) \stackrel{\text{def}}{=} level\ cut_r$
- Argue no arc is overused by more than factor 2 in expectation?

Caution: LP has a $\Omega(\log k)$ integrality gap

Level cuts of Calinescu, Karloff & Rabani

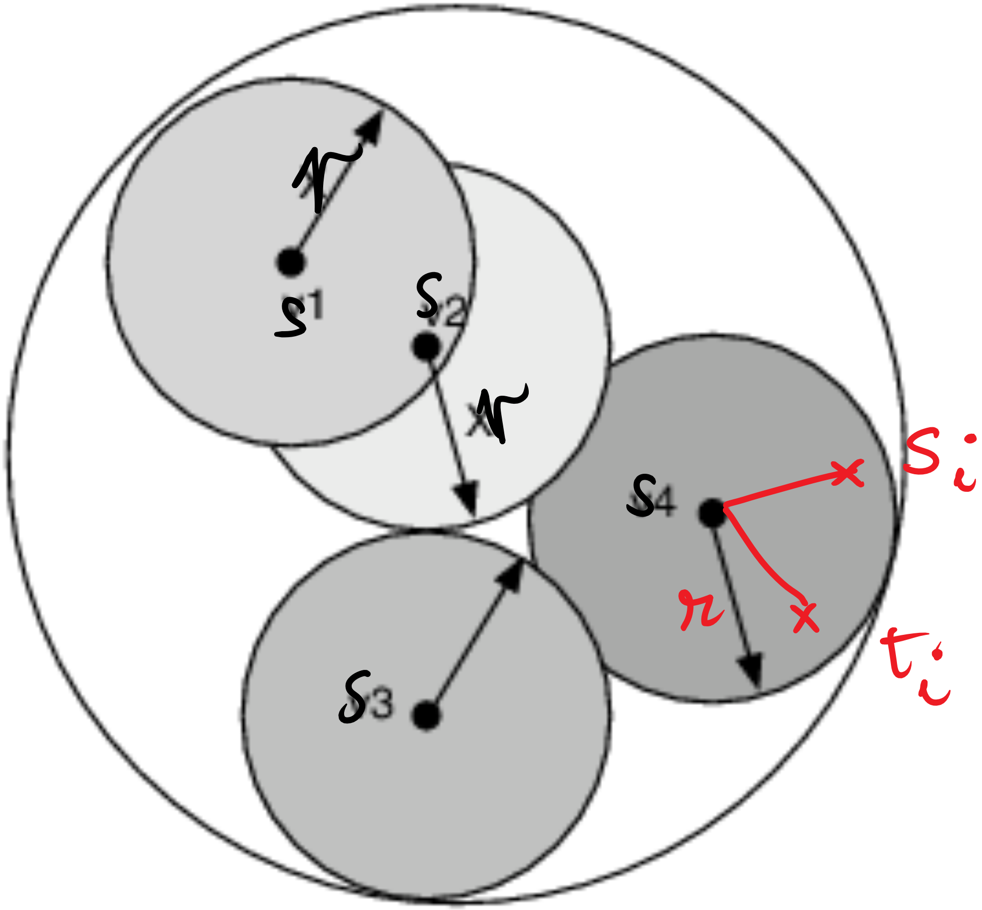
- Sort source-sink pairs in random order
- Pick random cutting radius $r \in (0,1)$
- In sorted order, cut edges leaving the ball $B(s_i, r)$ of nodes contained within a radius of r around the current source s_i with respect to the distances x_{uv} on edges $uv \in E(G)$
- "Protect" edges both of whose ends are contained in earlier balls from being cut later
- Output the unprotected parts $\cup_i \; \delta^u\big(B(s_i, r)\big) \stackrel{\text{def}}{=} level\ cut_r$ for a random r

Level cuts of Calinescu, Karloff & Rabani



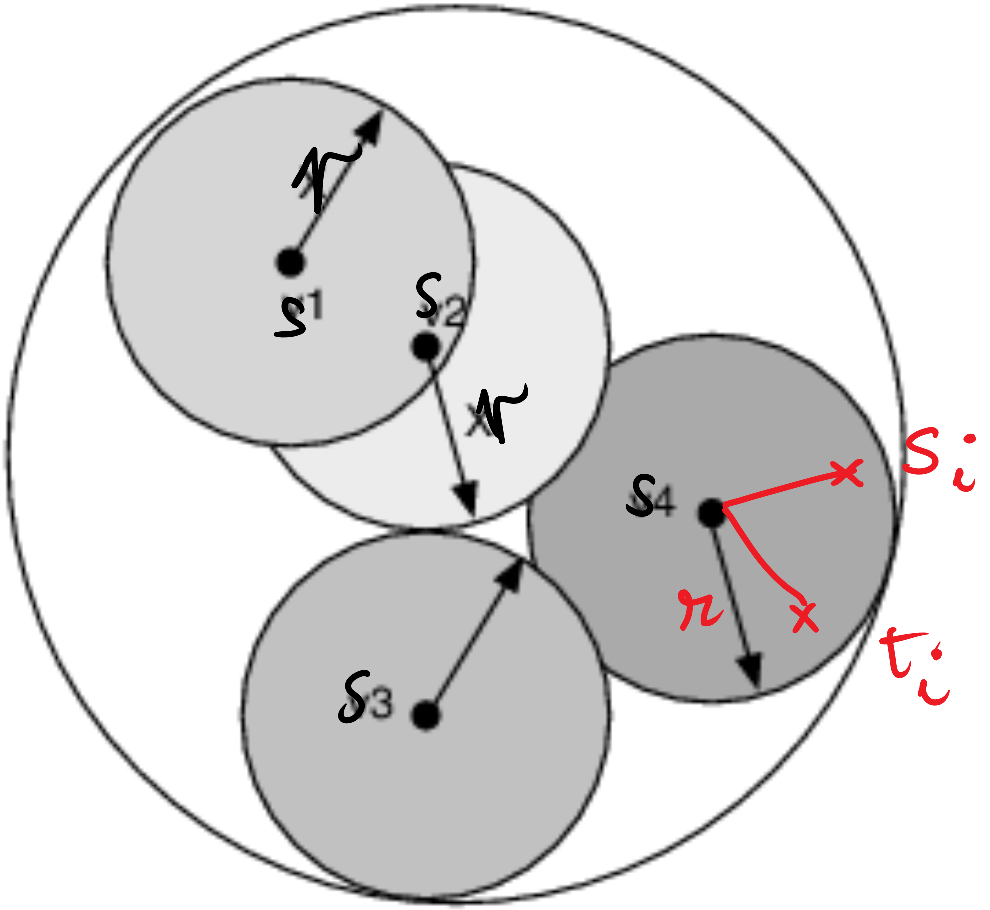
Level cuts of Calinescu, Karloff & Rabani

Is the solution feasible?



Level cuts of Calinescu, Karloff & Rabani

Reduce cutting radius to half



$d(s_i, t_i) \leq 2r$
< 1

Level cuts of Calinescu, Karloff & Rabani

- Sort source-sink pairs in random order
- Pick random cutting radius $r \in \left(0, \frac{1}{2}\right)$
- In sorted order, cut edges leaving the ball $B(s_i, r)$ of nodes contained within a radius of r around the current source s_i with respect to the distances x_{uv} on edges $uv \in E(G)$
- "Protect" edges both of whose ends are contained in earlier balls from being cut later
- Output the unprotected parts $\cup_i \; \delta^u\left(B(s_i, r)\right) \stackrel{\text{def}}{=} level\ cut_r$ for a random r

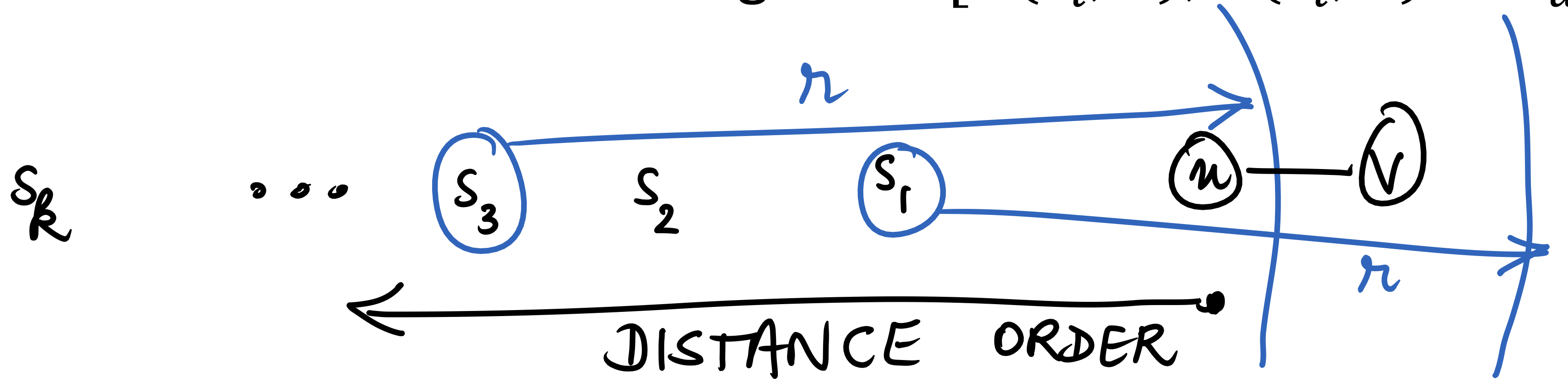
A 4 ln k approximation algorithm

Fix an edge uv. Order sources so that

$$d(s_1, u) \leq d(s_2, u) \leq \cdots \leq d(s_k, u)$$

When does s_i cut the edge from the u side?

- When no other s_j for $j < i$ occurs before it in the random order
- And when uv lies in the correct range: $r \in [d(s_i, u), d(s_i, u) + x_{uv}]$



A 4 ln k approximation algorithm

When does s_i cut the edge from the u side?

- When no other s_j for $j < i$ occurs before it in the random order

  probability $\leq \frac{1}{i}$

- And when uv lies in the correct range: $\mathrm{r} \in [d(s_i, u), d(s_i, u) + x_{uv}]$

  probability $\leq \frac{x_{uv}}{\frac{1}{2}}$

$$P(uv \in cut) = \sum_i P(uv\ cut\ by\ s_i) \leq 2 \sum_i \frac{1}{i} 2x_{uv} \leq 4 \ln k$$

A 4 ln k approximation algorithm

For the CKR cutting algorithm,

$$P(uv \in cut) \leq 4 \ln k$$

Theorem (CKR): Expected cost of output multicut is $4 \ln k \; z_{UMC}$

Summary

- If you need to cut a graph, write a distance based linear program to round
  - Minimum s-t cut in digraphs (folklore)
  - Multiway-cuts in undirected graphs (folklore)
  - Multiway-cuts in digraphs (Chekuri & Madan)
  - Multicuts in undirected graphs (Calinescu, Karloff & Rabani)