Integer Linear Programming and Bin Packing in fixed Dimension

Thomas Rothvoss

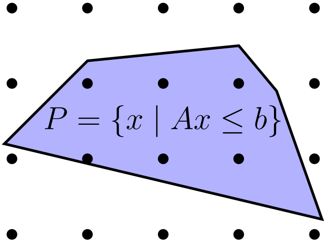
10th Cargèse Workshop on
Combinatorial Optimization (2019)

UNIVERSITY *of* WASHINGTON

Motivation and Outline

Integer Linear Programming

$$\max\{c^T x \mid Ax \leq b, x \in \mathbb{Z}^n\}$$



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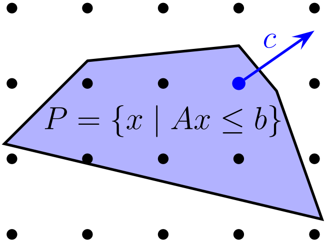

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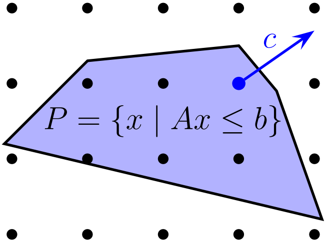

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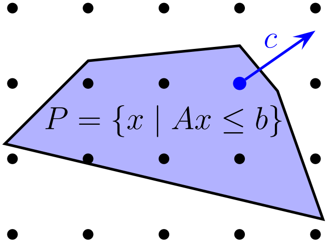
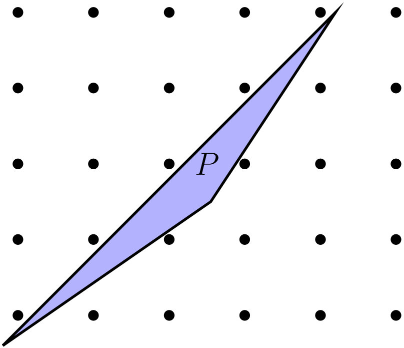


- **Part I:** Solve ILPs in time $f(n) \cdot \text{poly(input length)}$ [Lenstra '83], [Lenstra, Lenstra, Lovász '82]
- **Part II:** Solve **Bin Packing** with $O(1)$ different item types in poly-time [Goemans, R.' 14]

Part I

Solving ILPs in fixed dimension

Recursive approach for ILP



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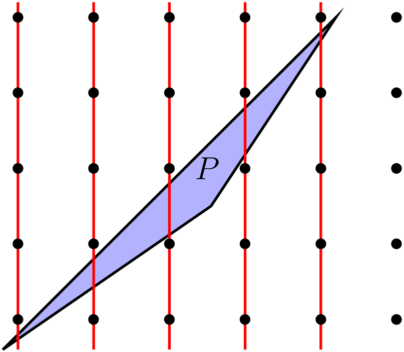

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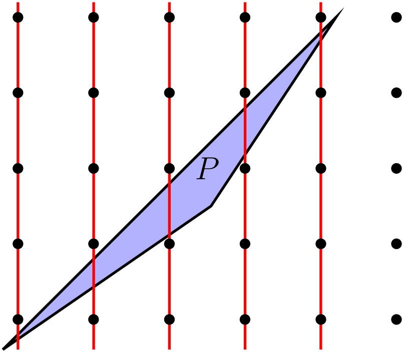

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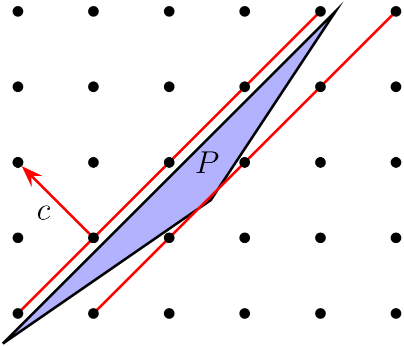
- **Idea 2:** Branch on general direction $c \in \mathbb{Z}^n$

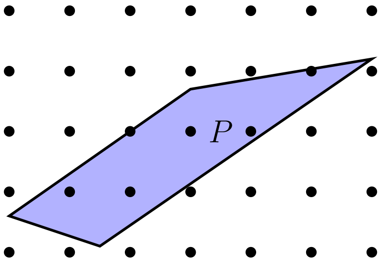


The flatness theorem

Theorem (Khintchine 1948, Lenstra 1983)

For polytope $P \subseteq \mathbb{R}^n$ in polynomial time one can find

- either a point $x \in P$
- a direction $c \in \mathbb{Z}^n$ with $\max\{\langle c, x - y\rangle \mid x, y \in P\} \leq 2^{O(n^2)}$



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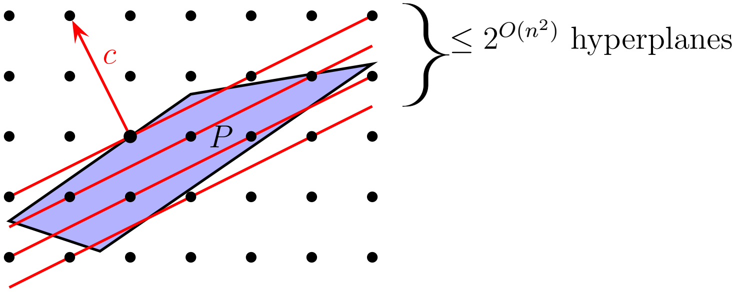

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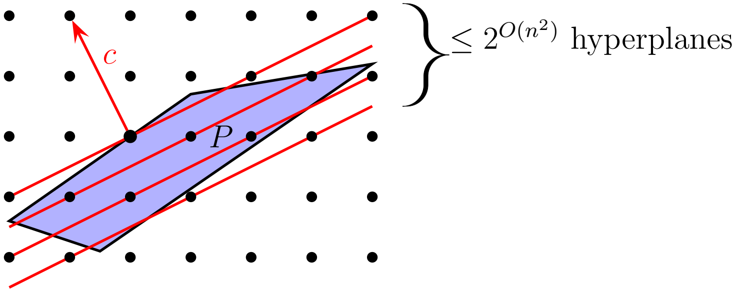
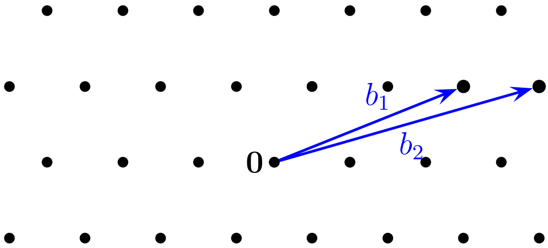


- Best current non-algo bounds: $\Omega(n) \leq .. \leq \tilde{O}(n^{4/3})$

Lattices

A **lattice** is $\Lambda = \{\sum_{i=1}^{n} \lambda_i \cdot b_i \mid \lambda_i \in \mathbb{Z}\}$ where $b_1, \ldots, b_n \in \mathbb{R}^n$ are linearly independent vectors



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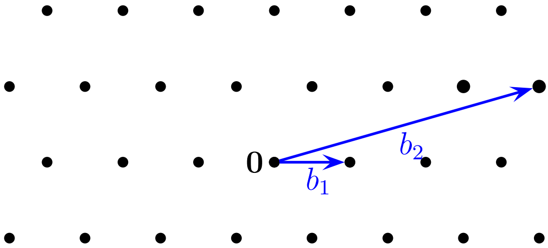

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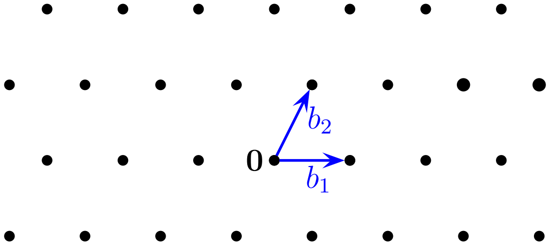

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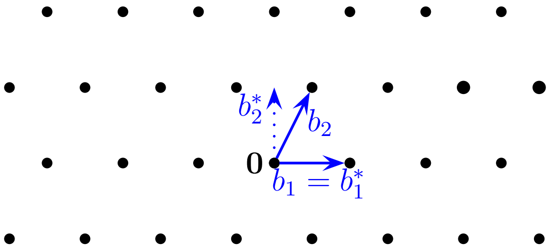


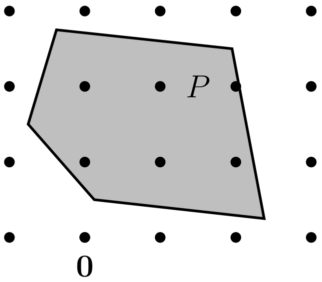
Theorem (Lenstra, Lenstra, Lovász '82)

In poly-time one can find a basis $b_1, \ldots, b_n$ so that the **orthogonality defect** is

$$\frac{\prod_{i=1}^n \|b_i\|_2}{\prod_{i=1}^n \|b_i^*\|_2} \leq 2^{n^2/2}$$

- Here $b_1^*, \ldots, b_n^*$ is the **Gram-Schmidt orthogonalization**.

Proof of Flatness Theorem



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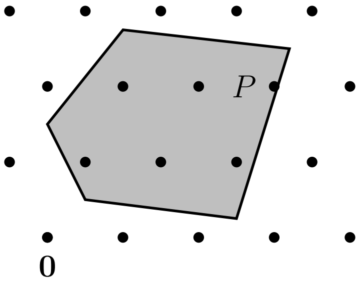

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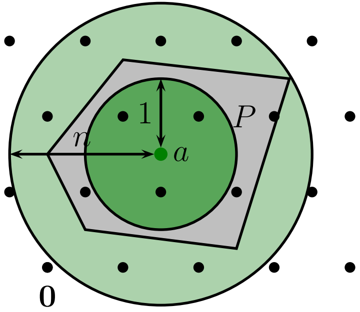

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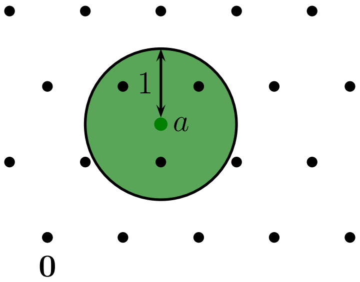

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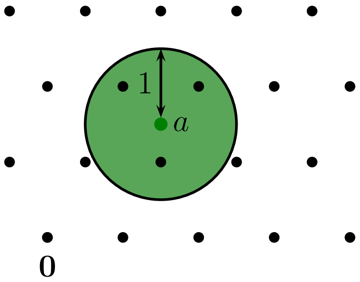

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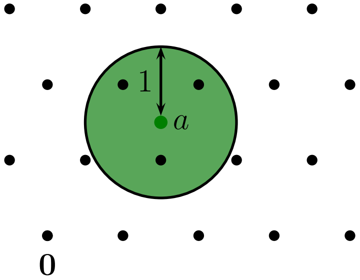

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Case $\|b_n\|_2 \leq \frac{1}{n}$**:**

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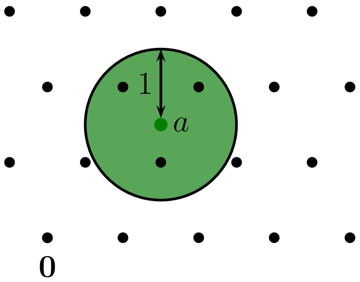

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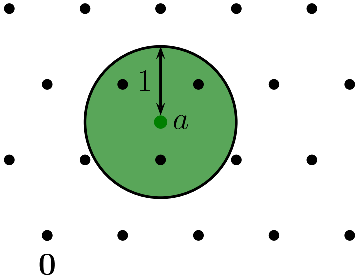

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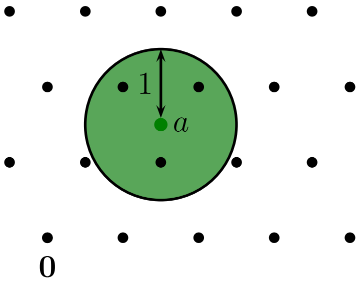

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- Return $\sum_{i=1}^n \lceil \lambda_i \rfloor \cdot b_i \in P$

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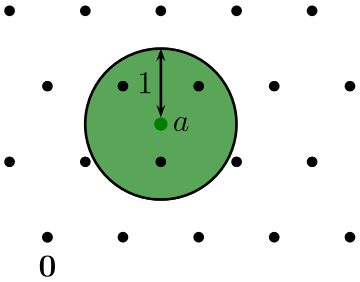

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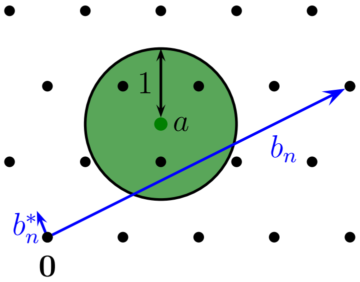

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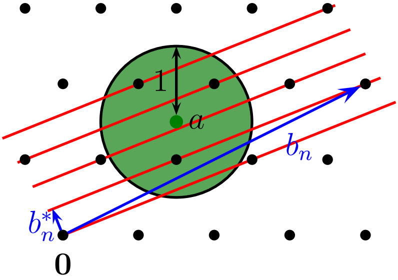

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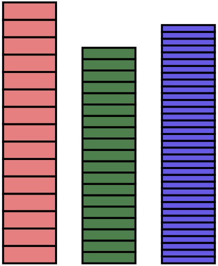
- Then $\|b_n^*\|_2 \geq 2^{-n^2/2}\|b_n\|_2 \geq 2^{-\Theta(n^2)}$
- Use $c := \frac{b_n^*}{\|b_n^*\|_2^2}$. Hyperplanes intersect $B(a,1)$ at most $2^{O(n^2)}$ times

Part II

Solving Bin Packing with a fixed number of item types

Bin Packing / Cutting Stock

Input:



item 1 $\cdots$ item d

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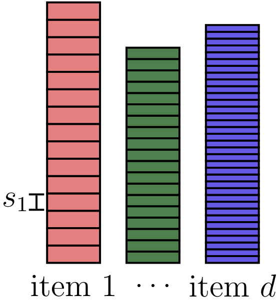

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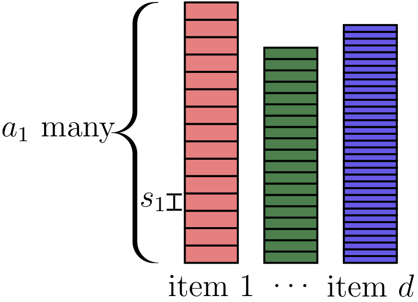

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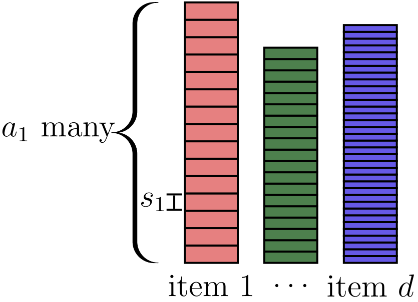

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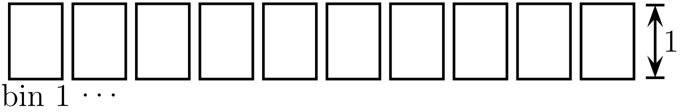

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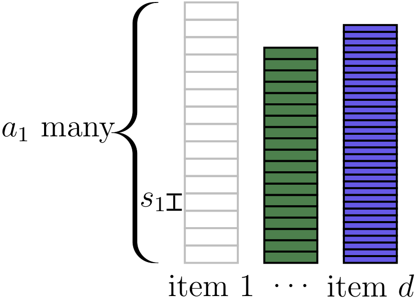

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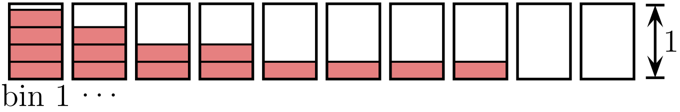

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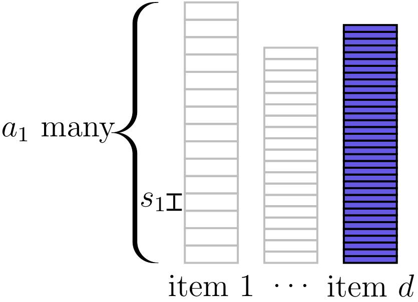

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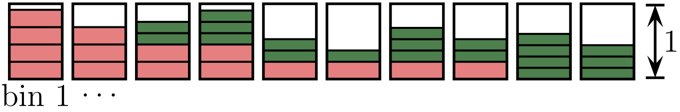

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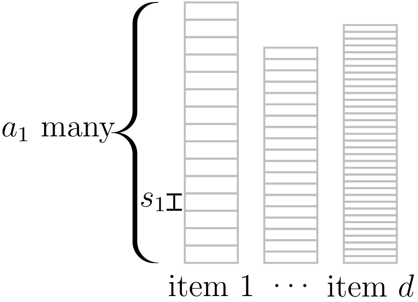

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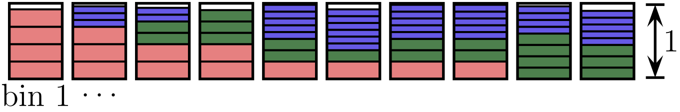

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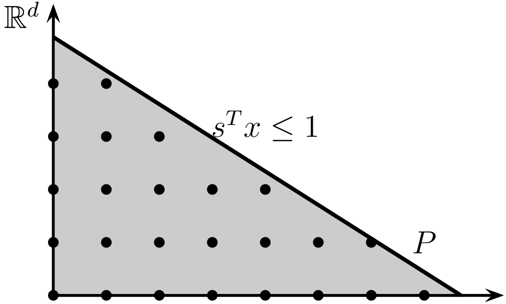
Open problem [MSS'97, ES'06, F'07]

Solvable in poly-time for $d = 3$?

A geometric view

- Define $P = \{x \in \mathbb{R}^d_{\geq 0} \mid s^T x \leq 1\}$

● a



A geometric view

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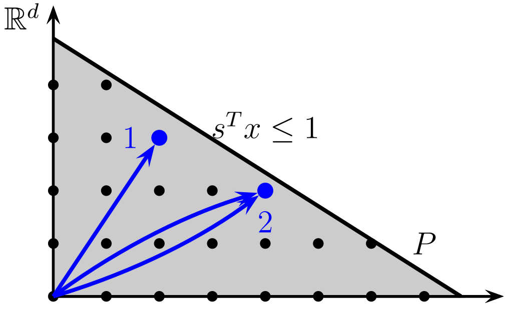

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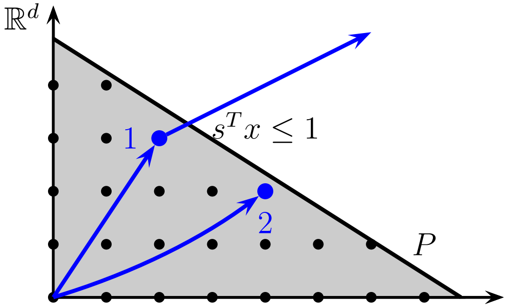

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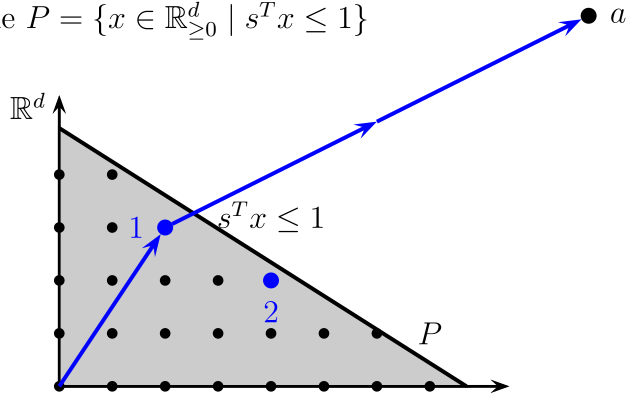

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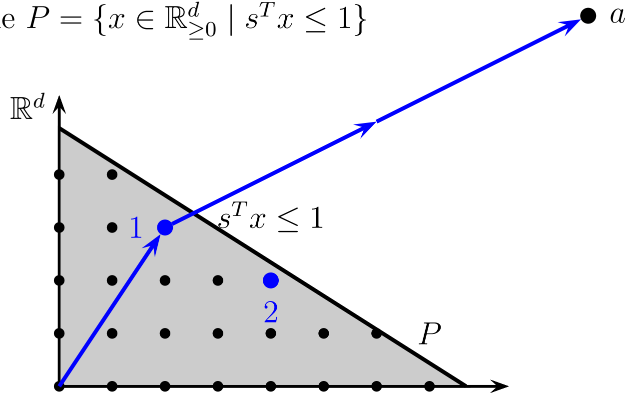

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- Points in P **exponentially** many

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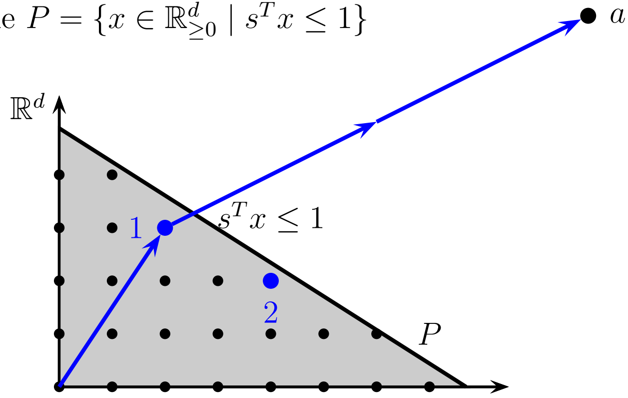

Problems:

- Points in P **exponentially** many
- Weights can be **exponential**

Main results

Theorem (Goemans, R. '13)

Bin Packing with $d = O(1)$ item sizes can be solved in **poly-time**.

Solves question by

- [McCormick, Smallwood, Spieksma '97]: "*might be* **NP**-*hard for* $d = 3$"
- [Eisenbrand & Shmonin '06]
- [Filippi '07]: "*hard open problem for general* d"

Main results (2)

- **Def.:** $\text{int.cone}(X) := \{\sum_{x \in X} \lambda_x \cdot x \mid \lambda_x \in \mathbb{Z}_{\geq 0}\}$

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For **fixed-dim. polytopes** $P, Q \subseteq \mathbb{R}^d$, testing

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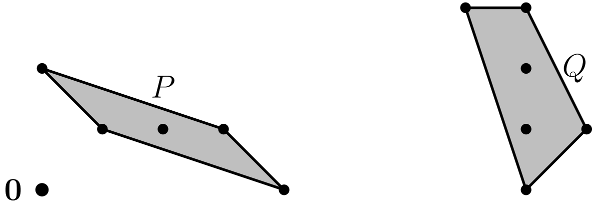
is doable in **poly-time** (actually inputlength$^{2^{O(d)}}$).



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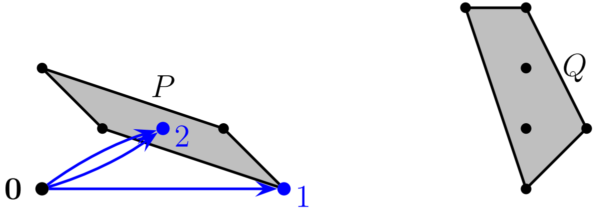

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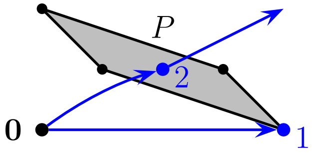

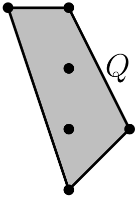

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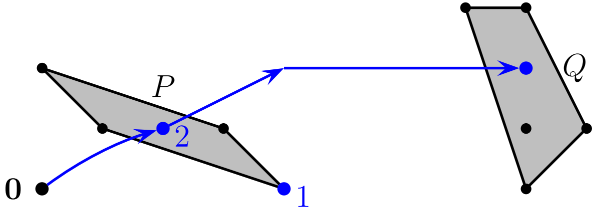

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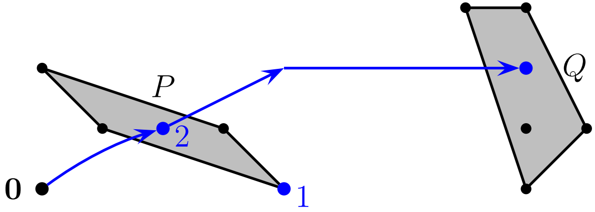


- For Bin Packing:
  $P := \{\binom{x}{1} \mid s^T x \leq 1,\ x \geq \mathbf{0}\}$ and $Q := \{\binom{a}{OPT}\}$

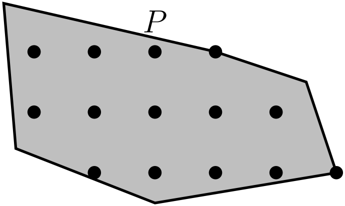
Integer conic combinations

Theorem (Eisenbrand & Shmonin '06)

If $P \subseteq \mathbb{R}^d$ convex, then any integer conic combination

$$a = \sum_{x \in P \cap \mathbb{Z}^d} \lambda_x \cdot x$$

needs at most 2^d points.



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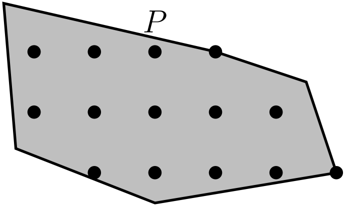

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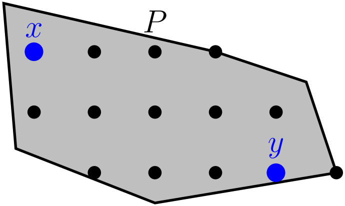

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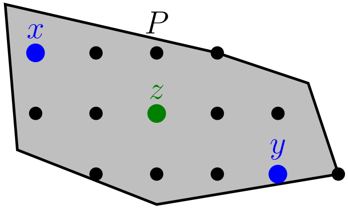

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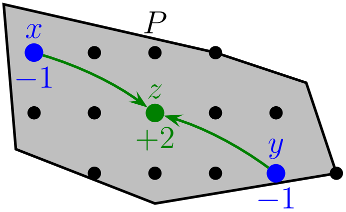

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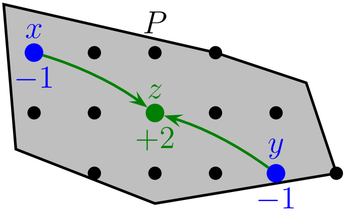

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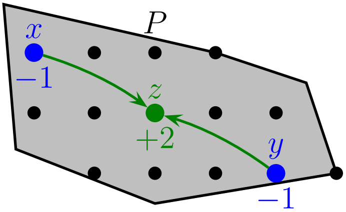
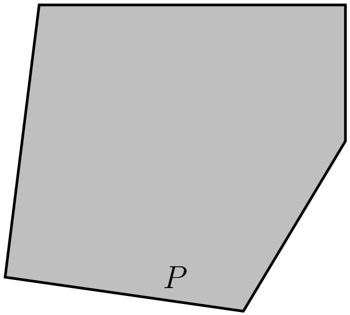


- **Problem:** Still don't know which points to take!

Redistributing weight



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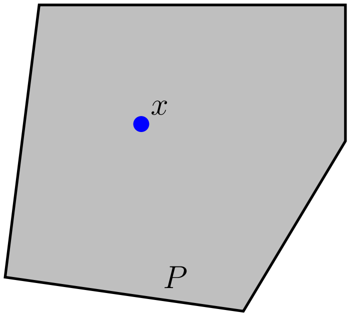
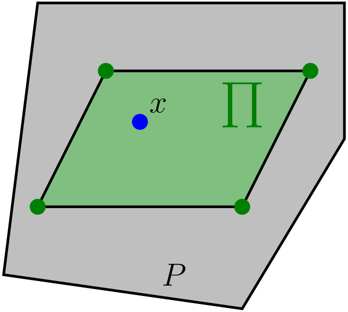


Redistributing weight

- Consider **parallelepiped** $\Pi \ni x$ with integral vertices



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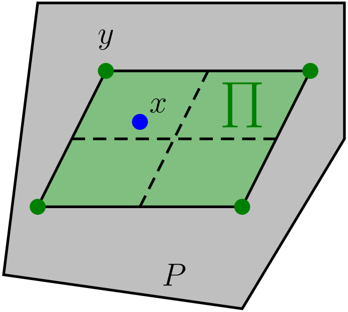

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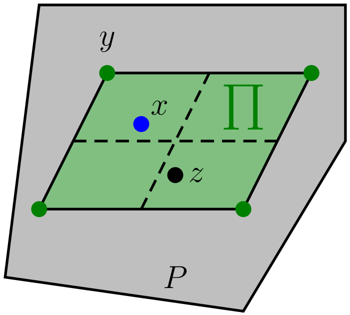

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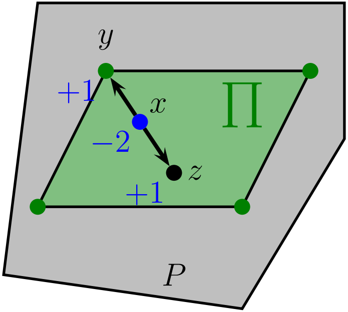

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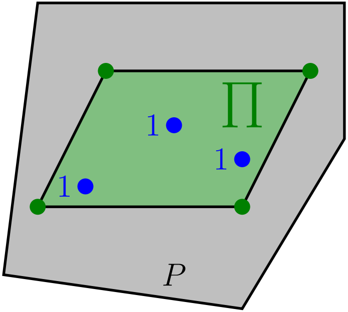

Redistributing weight

Lemma

For x in parallelepiped Π and $\lambda_x \in \mathbb{N}$, one can write

$$\lambda_x x = \text{int.cone(vertices of } \Pi) + \sum \text{of } 2^d \text{ points in } \Pi \cap \mathbb{Z}^d$$

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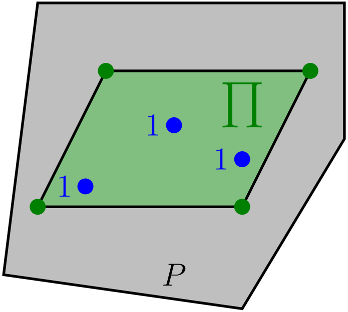
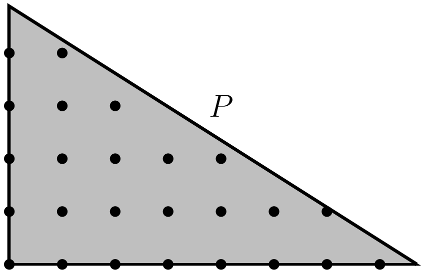


Covering a polytope with parallelepipeds

Lemma

For fixed-dim $P \subseteq \mathbb{R}^d$, we can cover $P \cap \mathbb{Z}^d$ with **poly-many parallelepipeds** (with int. vertices and $\subseteq P$).



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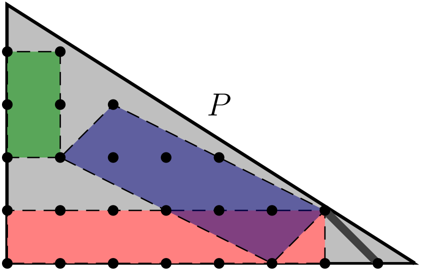
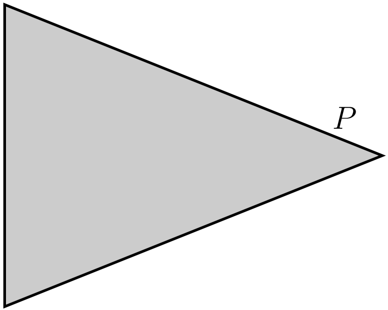


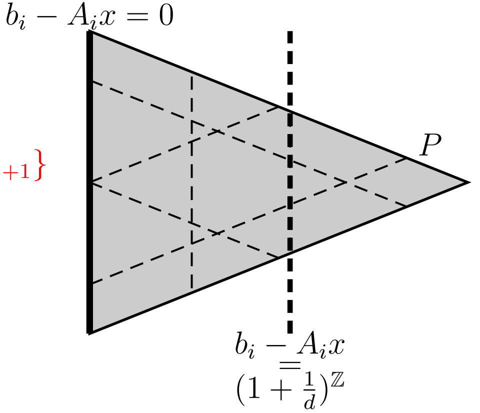
Covering a polytope w. parallelep. (2)



Covering a polytope w. parallelep. (2)

- Split $P = \{x \mid Ax \leq b\}$ into poly many **cells**

$C = \{x \mid \alpha_{j(i)} \leq A_i x \leq \alpha_{j(i)+1}\}$



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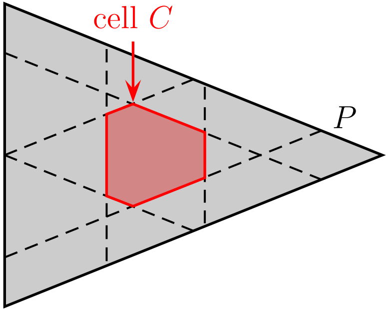

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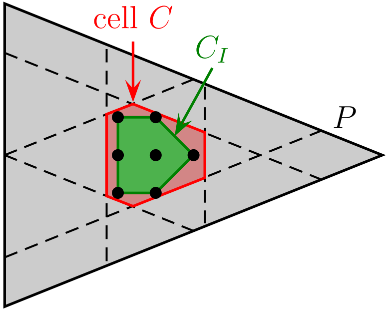

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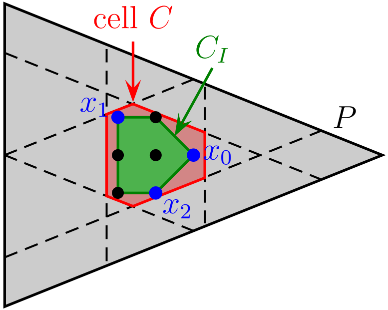

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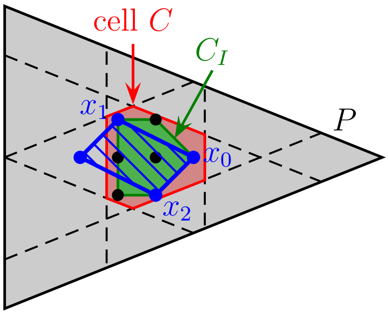

The algorithm

- **Input:** polytopes P, Q in ineq. description
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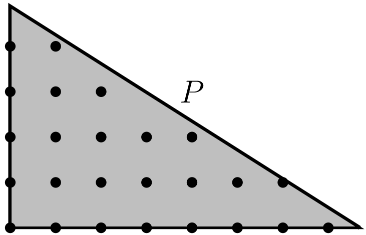

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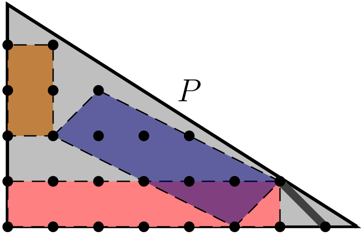

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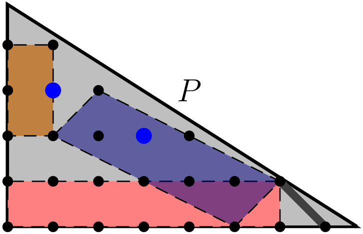

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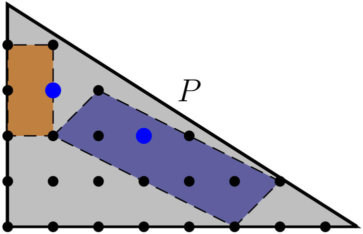

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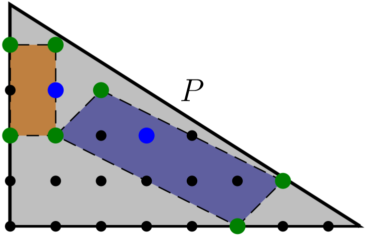

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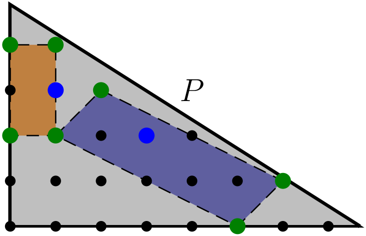
(3) Solve ILP with $2^{O(d)}$ variables

$$\sum_{x \in X} \lambda_x \cdot x + \sum_{\leq 2^{2d}} \sum_{x \in P \cap \mathbb{Z}^d} 1 \cdot x \in Q$$



The algorithm

- **Input:** polytopes P, Q in ineq. description
- **Output:** Coefficients for $\text{int.cone}(P \cap \mathbb{Z}^d) \cap Q \neq \emptyset$

Algorithm:

(1) Compute poly many parallelepipeds covering P

(2) Guess the 2^d parallelepipeds containing solution
$\rightarrow X :=$ vertices

(3) Solve ILP with $2^{O(d)}$ variables

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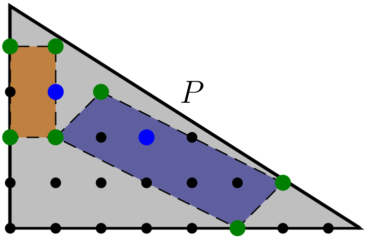
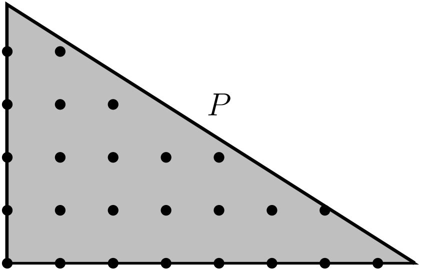
variables



A Structure Theorem

Structure Theorem

For polytope P



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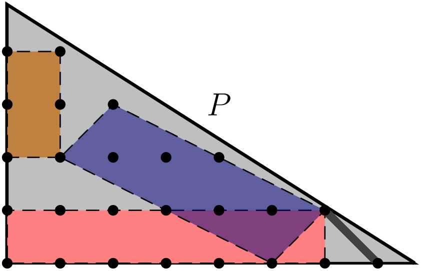

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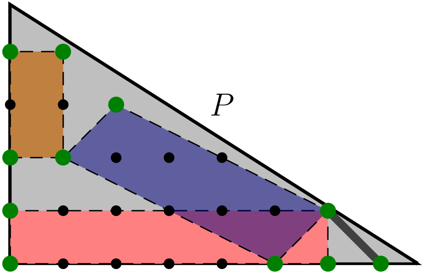
For polytope P, $\exists$ poly-time comp. set $\boldsymbol{X} \subseteq P \cap \mathbb{Z}^d$



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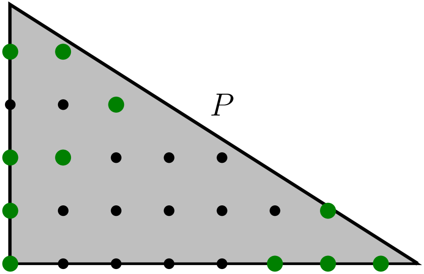

A Structure Theorem

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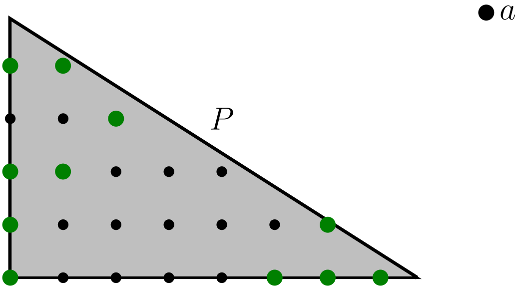
For polytope P, $\exists$ poly-time comp. set $\boldsymbol{X} \subseteq P \cap \mathbb{Z}^d$ s.t. **for all** $a \in \textbf{int.cone}(P \cap \mathbb{Z}^d)$ one can express

$$a = \text{int.cone}(2^{2d} \text{ points in } X) + \sum \text{of } 2^{2d} \text{ points in } P \cap \mathbb{Z}^d$$



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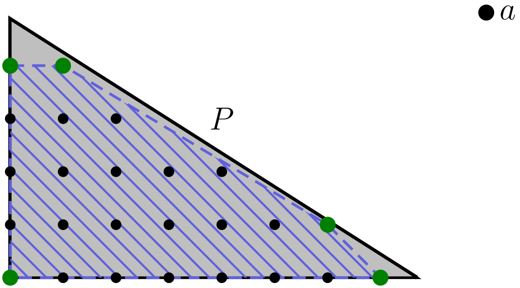

- More recent: $a = \text{int.cone}(\text{vert}(P_I)) + \sum$ of $2^{O(d)}$ points with weights $\leq 2^{2^{O(d)}}$ [Jansen, Klein '16]

Open questions

- Is there a $2^{O(n)} \cdot \text{poly}(\text{input})$ algorithm for **Integer Linear Programming**?
- Is there a poly(n)-factor approximation algorithm for **Shortest Vector Problem** in a Lattice?
(distinguishing $\geq \Theta(\sqrt{n}) \cdot L$ from $\leq L$ is in $\mathbf{NP} \cap \mathbf{coNP}$)
- Can one find a lattice basis with orthogonality defect at most $n^{O(n)}$ in poly-time?
- Can **Bin Packing** be solved in $\mathbf{(input\ length)^{poly(\mathit{d})}}$ time? maybe even in $f(d) \cdot \text{poly}(\text{input length})$?

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Thanks for your attention