Schwiegelshohn's Proof of the Kawaguchi-Kyan Bound

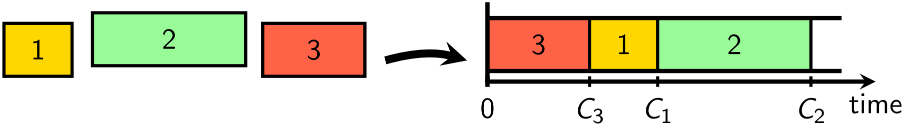
Martin Skutella

TU Berlin

Single Machine Scheduling to Minimize $\sum w_j C_j$

Given: n jobs $j = 1, \ldots, n$, processing times $p_j > 0$, weights $w_j > 0$

Task: schedule jobs on a single machine; minimize $\sum_j w_j C_j$



Weighted Shortest Processing Time (WSPT) rule:

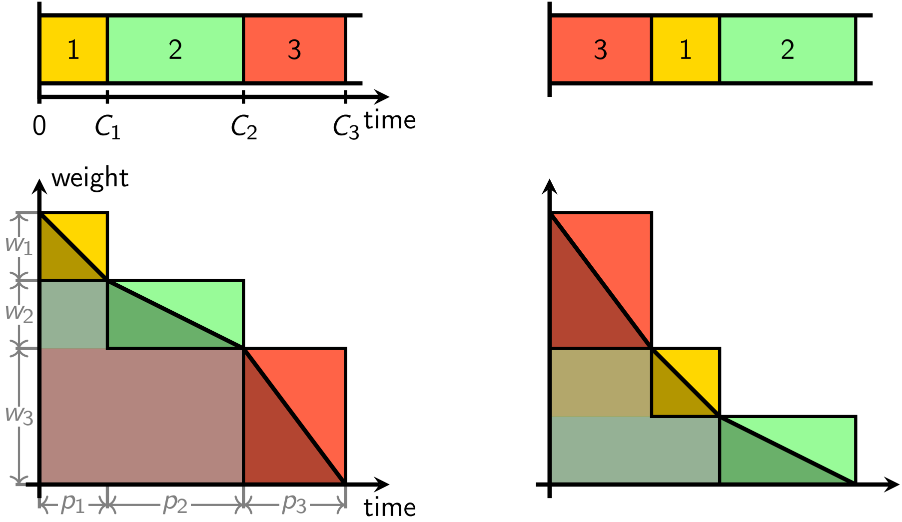
Theorem (Smith 1956).

Sequencing jobs in order of non-increasing ratios w_j / p_j is optimal.

"Photographer's Rule"

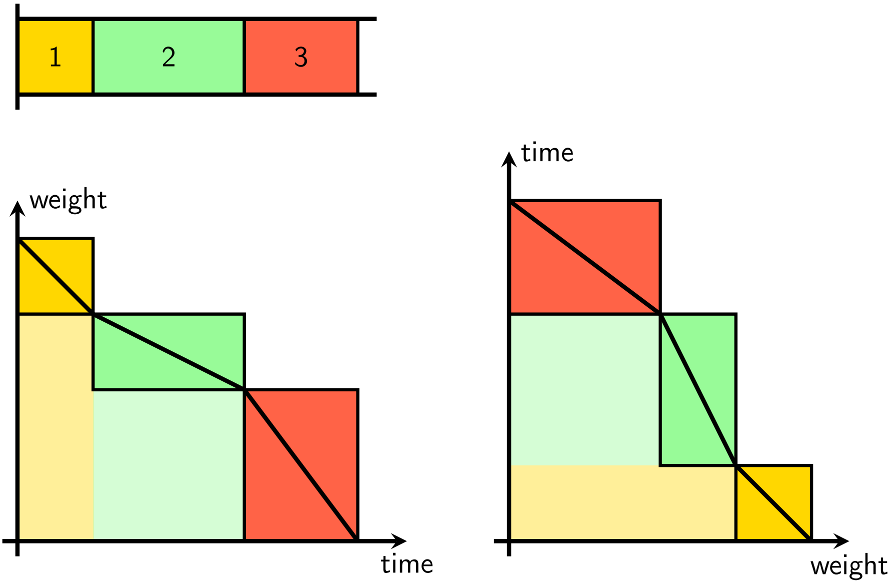
Two-Dimensional Gantt Charts

Eastman, Even & Isaacs 1964; Goemans & Williamson 2000



w_j/p_j = diagonal slope of rectangle representing job j

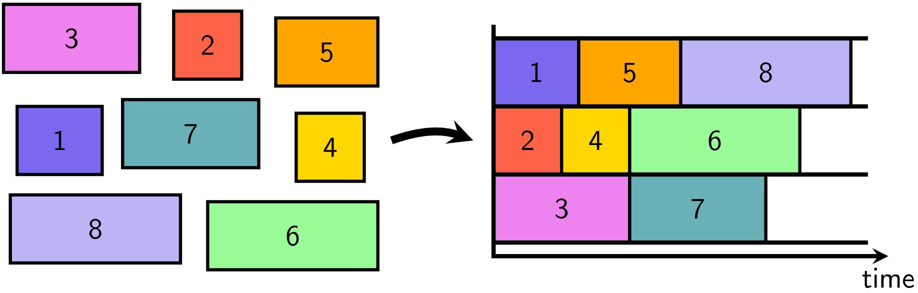
Swap Weights and Processing Times



Parallel Machine Scheduling to Minimize $\sum w_j C_j$

Given: n jobs $j = 1, \ldots, n$, processing times $p_j > 0$, weights $w_j > 0$

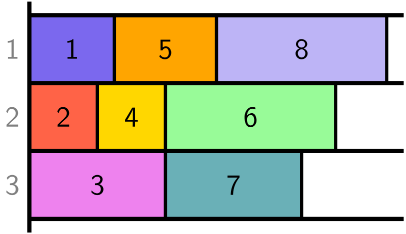
Task: schedule jobs on m parallel machines; minimize $\sum_j w_j C_j$



- weakly NP-hard for two machines (Bruno, Coffman & Sethi 1974)
- strongly NP-hard if m part of input (Garey & Johnson, problem SS13)
- PTAS (Sk. & Woeginger 2000)

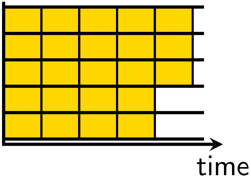
List Scheduling in Order of Non-Increasing w_j/p_j

$w_1/p_1 \geq w_2/p_2 \geq \cdots \geq w_n/p_n$



Theorem (Conway, Maxwell & Miller 1967).

Optimal if $w_j = 1$ for all j (or: $p_j = 1$ for all j).



Theorem (Kawaguchi & Kyan 1986).

Tight performance ratio: $\frac{1+\sqrt{2}}{2} \approx 1,207\ldots$

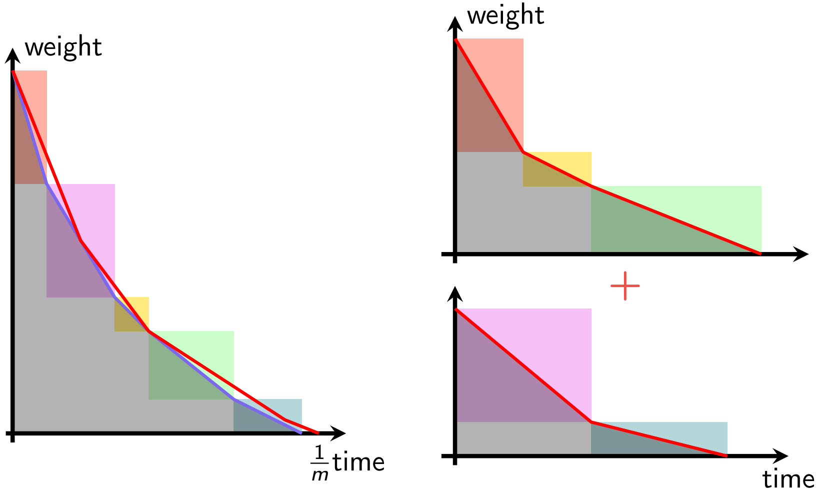
Outline

1. WSPT has performance ratio $\leq 3/2$
2. WSPT has performance ratio exactly $\frac{1}{2}(1+\sqrt{2}) \approx 1{,}207\ldots$
3. WSEPT for stochastic scheduling
4. Open problem

Fast Single Machine Lower Bound

Lemma (Eastman, Even & Isaacs 1964).

$$\frac{1}{m}\left(\mathrm{OPT}_1 - \frac{1}{2}\sum_j w_j p_j\right) \;\leq\; \mathrm{OPT}_m - \frac{1}{2}\sum_j w_j p_j$$

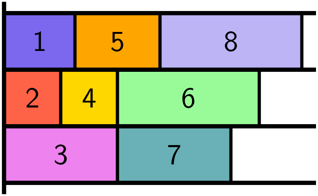


WSPT has Performance Ratio $\leq 3/2$

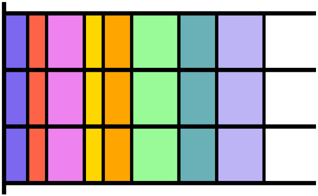
Lemma (Eastman, Even & Isaacs 1964).

$$\frac{1}{m}\left(\mathrm{OPT}_1 - \frac{1}{2}\sum_j w_j p_j\right) \leq \mathrm{OPT}_m - \frac{1}{2}\sum_j w_j p_j$$

WSPT



OPT_1/m



WSPT start times $\leq$ single machine start times

Thus:

$$\mathrm{WSPT}_m \leq \frac{1}{m}\left(\mathrm{OPT}_1 - \frac{1}{2}\sum_j w_j p_j\right) + \sum_j w_j p_j$$

$$\leq \mathrm{OPT}_m + \frac{1}{2}\sum_j w_j p_j \leq \frac{3}{2}\mathrm{OPT}_m$$

Schwiegelshohn's Proof of the Kawaguchi-Kyan Bound

Theorem (Kawaguchi & Kyan 1986).

WSPT has performance ratio exactly $\frac{1+\sqrt{2}}{2} \approx 1,207\ldots$

Proof idea: explicit construction of worst-case instance (for $m \to \infty$)

Refined: exact performance ratio for each fixed m (Jäger & Sk. 2018)

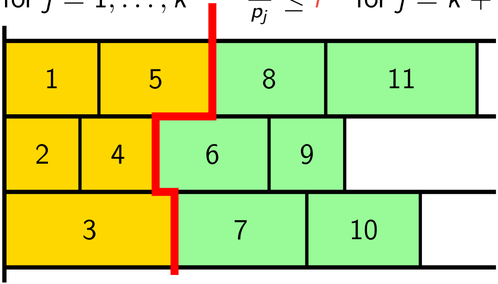
Sequence of reductions to worst-case instances with:

1. $w_j = p_j$ for all j
2. at most $m - 1$ large jobs and many tiny jobs
3. all but one large job are extra-large
4. all XL jobs have same size

First Reduction: $w_j = p_j \; \forall j$

$\frac{w_j}{p_j} \geq R \quad \text{for } j = 1, \ldots, k \qquad \frac{w_j}{p_j} \leq r \quad \text{for } j = k+1, \ldots, n$

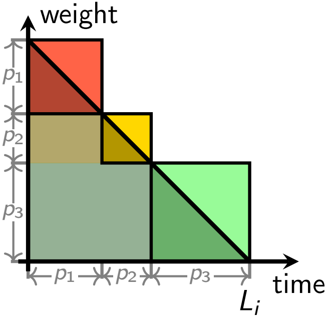
$R > r$



$$\sum_{j=1}^{n} w_j C_j = \frac{r}{R} \sum_{j=1}^{k} w_j C_j + \sum_{j=k+1}^{n} w_j C_j + \left(1 - \frac{r}{R}\right) \sum_{j=1}^{k} w_j C_j$$

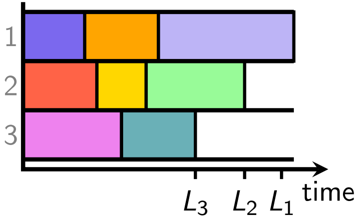
$$\Longrightarrow \quad \frac{\text{WSPT}}{\text{OPT}} = \frac{A_{\text{WSPT}} + B_{WSPT}}{A_{\text{OPT}} + B_{\text{OPT}}} \leq \max\left\{ \frac{A_{\text{WSPT}}}{A_{\text{OPT}}}, \frac{B_{WSPT}}{B_{\text{OPT}}} \right\}$$

Objective Function in Terms of Machine Loads (for $w_j = p_j$)



one machine i:

$$\sum_{j \to i} p_j C_j = \frac{1}{2}\left(\sum_{j \to i} p_j\right)^2 + \frac{1}{2}\sum_{j \to i} {p_j}^2$$



m-machine schedule:

$$\sum_{j=1}^{n} p_j C_j = \frac{1}{2}\sum_{i=1}^{m} {L_i}^2 + \frac{1}{2}\sum_{j=1}^{n} {p_j}^2$$

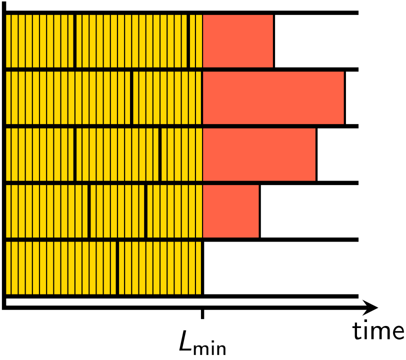
notice:

- $\sum_i L_i = \sum_j p_j$ (fixed)
- $\sum_i {L_i}^2$ minimal if $L_1 = \cdots = L_m$

Second Reduction: Large Jobs and Sand

$$\sum_j p_j C_j = \tfrac{1}{2} \sum_i {L_i}^2 + \tfrac{1}{2} \sum_j {p_j}^2$$

WSPT schedule



WSPT:

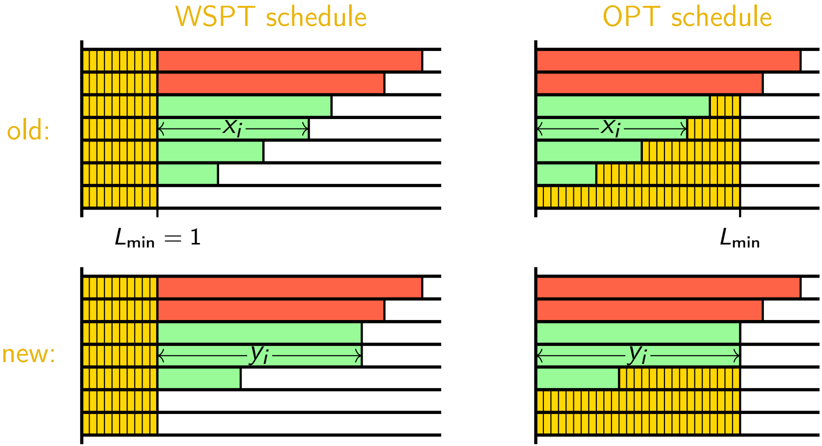
- $\sum_i {L_i}^2$ remains unchanged
- $\sum_j {p_j}^2$ decreases by $\delta \geq 0$

OPT:

- $\sum_i {L_i}^2$ unchanged or decreases
- $\sum_j {p_j}^2$ decreases by δ

$\implies \dfrac{\text{WSPT}}{\text{OPT}}$ unchanged or increases

Third Reduction: Make Large Jobs Extra-Large

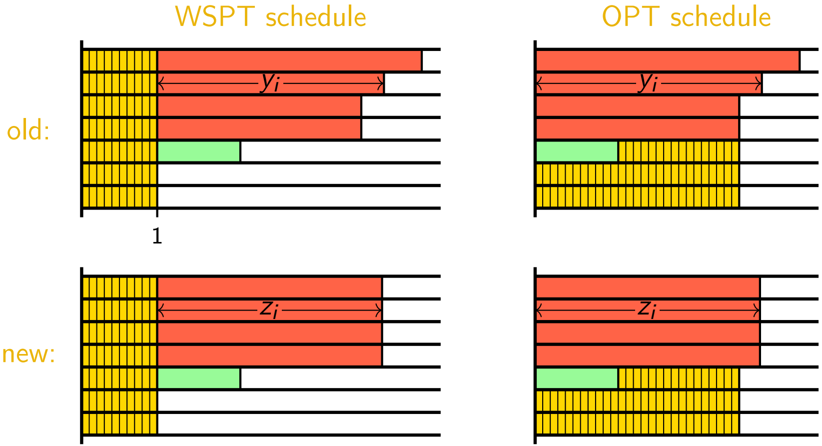


Increase in objective:

$$\frac{1}{2}\sum_i \big((1+y_i)^2 + {y_i}^2 - (1+x_i)^2 - {x_i}^2\big) \qquad \frac{1}{2}\sum_i \big({y_i}^2 - {x_i}^2\big) \geq 0$$

$$= \sum_i \big({y_i}^2 - {x_i}^2\big) \qquad \text{as } \sum_i x_i = \sum_i y_i$$

Fourth Reduction: All XL Jobs of Same Size

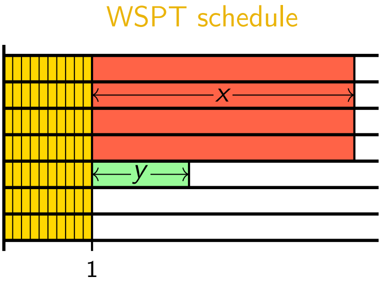


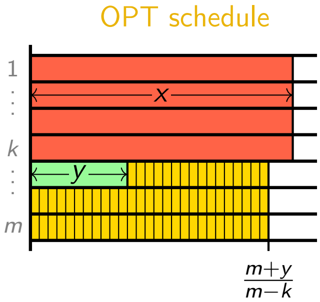
Increase in objective:

$$\frac{1}{2}\sum_i\big((1+z_i)^2+{z_i}^2-(1+y_i)^2-{y_i}^2\big) \qquad \sum_i\big({z_i}^2-{y_i}^2\big)\leq 0$$

$$=\sum_i\big({z_i}^2-{y_i}^2\big) \qquad \text{as } \sum_i x_i=\sum_i y_i$$

Analyzing the Performance Ratio





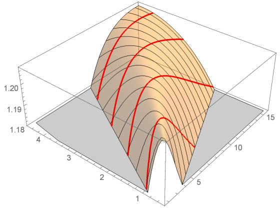
$$\text{WSPT} = \frac{m}{2} + k \cdot x(1+x) + y(1+y) \qquad \text{OPT} = k \cdot x^2 + \frac{(m+y)^2}{2(m-k)} + \frac{y^2}{2}$$

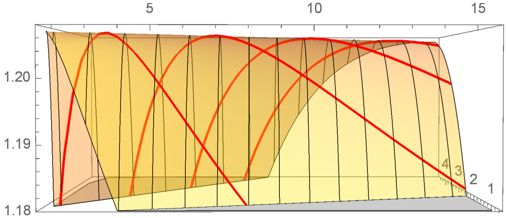
$$\frac{\text{WSPT}}{\text{OPT}} = \frac{(m-k)(2kx^2 + 2kx + 2y^2 + 2y + m)}{(m-k)(2kx^2 + y^2) + (y+m)^2}$$

Observation: maximum at $y = 0$ and $x = \dfrac{m}{\sqrt{k(2m-k)} - k}$

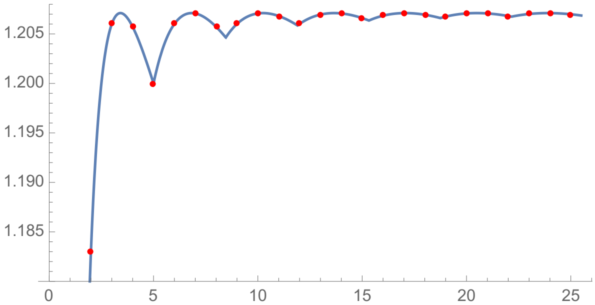
Worst-Case Instance

worst-case performance ratio: $\max_k \left(1 - \frac{k}{2m} + \sqrt{\frac{k}{2m}\left(1 - \frac{k}{2m}\right)}\right)$



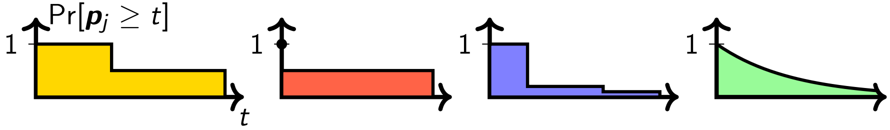


Observation: maximum at $k = \left\lfloor \left(1 - \frac{1}{2}\sqrt{2}\right) m \right\rfloor$.



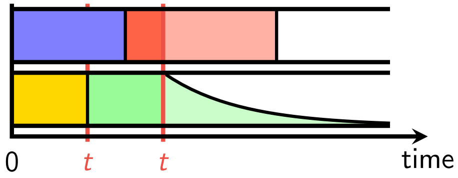
Stochastic Scheduling

Given: distributions of independent random processing times $\boldsymbol{p}_j \geq 0$



Task: find scheduling policy minimizing $\mathrm{E}\left[\sum w_j C_j\right]$

- scheduling policy must be nonanticipatory, i.e., decision made at time t may only depend on the information known at time t



Weighted Shortest Expected Processing Time (WSEPT)

WSEPT Rule

List scheduling in order of non-increasing $w_j / \mathrm{E}[\boldsymbol{p}_j]$.

- WSEPT is optimal for single machine (Rothkopf 1966)
- WSEPT has performance ratio $1 + \frac{1}{2}(1 + \Delta)$ with $\Delta \geq \frac{\mathrm{Var}[\boldsymbol{p}_j]}{\mathrm{E}[\boldsymbol{p}_j]^2}$ for all j. (Möhring, Schulz & Uetz 1999)
- WSEPT has no constant performance ratio. (Cheung, Fischer, Matuschke & Megow 2014; Im, Moseley & Pruhs 2015)
- WSEPT has performance ratio $1 + \frac{1}{2}(\sqrt{2} - 1)(1 + \Delta)$. (Jäger & Sk. 2018)

Open Problem

Online setting:

- jobs arrive one by one; must be immediately assigned to machines
- on each machine, assigned jobs are optimally sequenced (WSPT)

Algorithm MinIncrease

- assign job to machine minimizing increase of current objective value

Known results:

- MinIncrease has competitive ratio $\frac{3}{2} - \frac{1}{2m}$.
- If jobs arrive in order of non-increasing or non-decreasing w_j/p_j, then MinIncrease achieves competitive ratio $\frac{1}{2}(1+\sqrt{2})$.

Conjecture (Stougie 2017).

MinIncrease has competitive ratio $\frac{1}{2}(1+\sqrt{2})$.